POSITIVE PERIODIC SOLUTIONS OF A CLASS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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POSITIVE PERIODIC SOLUTIONS OF A CLASS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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The article considers the question of the existence of positive ω-periodic solutions of the nonlinear second-order equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = f(t, x), \tag{1}$$

where $p(t)$ and $q(t)$ are continuous ω-periodic functions. With respect to the function $f(t, x)$, it is assumed that it is continuous for $-\infty < t < \infty$, $x \geq 0$, ω-periodic in t, and nonnegative for nonnegative x. In addition, throughout the article it is assumed that the solutions of the linear equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{2}$$

do not oscillate on (t_0, ∞) (see ([1],[2])) and that

$$\int_0^\omega p(\tau)\, d\tau > 0.$$

In the theorems formulated below, the linear equations

$$\ddot{x} + p(t)\dot{x} + q(t)x - c_i(t)x = 0 \quad (i = 1, 2, 3, 4). \tag{3}$$

will appear. Without specifying this each time, we shall assume that the functions $c_i(t)$ $(i = 1, 2, 3, 4)$ are continuous, nonnegative, and ω-periodic.

1. **Theorem 1.** *Suppose the inequality*

$$f(t, x) \leq c_1(t)x + a \quad (x \geq 0,\ 0 \leq t \leq \omega) \tag{4}$$

is satisfied.

Suppose the zero solution of equation (3) $(i = 1)$ *is asymptotically stable.*

Then equation (1) has at least one nonnegative ω*-periodic solution.*

2. In the remainder of the article it is assumed that $f(t, 0) \equiv 0$ $(0 \leq t \leq \omega)$, and the question of the existence of nonzero periodic solutions is considered.

Theorem 2. *Suppose condition (4) is satisfied, and suppose that for small nonnegative* x

$$f(t, x) \geq c_2(t)x \quad (0 \leq t \leq \omega).$$

Finally, suppose the zero solution of equation (3) $(i = 2)$ *is unstable.*

Then equation (1) has at least one positive ω*-periodic solution.*

Theorem 3. *Suppose that, for small nonnegative* x*, the inequality*

$$f(t, x) \leq c_3(t)x \quad (0 \leq t \leq \omega),$$

and for all nonnegative x

$$f(t, x) \geq c_4(t)x - b \qquad (0 \leq t \leq \omega).$$

Finally, suppose that the zero solution of equation (3) $(i = 3)$ is asymptotically stable, while the zero solution of equation (3) $(i = 4)$ is unstable. Then equation (1) has at least one positive ω-periodic solution.

3. Following M. A. Krasnosel' skii ([3]), we shall call a function $f(t, x)$ **quasi-concave** if

$$f(t, \tau x) \geq \tau f(t, x) \qquad (0 \leq t \leq \omega,\ x \geq 0,\ 0 < \tau < 1).$$

Similarly, if

$$f(t, \tau x) \leq \tau f(t, x) \qquad (0 \leq t \leq \omega,\ x \geq 0,\ 0 < \tau < 1),$$

then we shall call the function $f(t, x)$ **quasi-convex**.

Theorem 4. Suppose that the function $f(t, x)$ is quasi-concave and that it is nondecreasing in the second variable for $x \geq 0$. Further, suppose that for some $0 \leq t_1 \leq \omega$ and $x > 0$

$$f(t_1, \tau x) > \tau f(t_1, x) \qquad (0 < \tau < 1).$$

Then, under the conditions of Theorems 1 and 2, equation (1) has a positive ω-periodic solution asymptotically stable in the sense of Lyapunov.

Theorem 5. Suppose that the function $f(t, x)$ is quasi-convex. Suppose that for some $0 \leq t_2 \leq \omega$ and $x > 0$

$$f(t_2, \tau x) < \tau f(t_2, x) \qquad (0 < \tau < 1).$$

Then, under the conditions of Theorem 3, equation (1) has an unstable positive ω-periodic solution.

4. A natural question arises about the number of ω-periodic solutions of equation (1). We give one result.

Theorem 6. Suppose that for all $0 \leq t \leq \omega$ the inequality

$$\int_t^{t+\omega} \exp\left[\int_0^s p(\tau)\, d\tau\right] q(s)\, ds \geq 0$$

is satisfied.

Then, under the conditions of Theorems 4 and 5, equation (1) has a unique positive ω-periodic solution and has no sign-changing ω-periodic solutions.

All the theorems formulated above are obtained by methods of cone theory (see ([3–5])).

5. The question of the stability of solutions of the linear equation (2) under various assumptions has been studied by many authors (see, for example, ([6–8])). Here we shall give criteria for stability and instability of solutions of equation (2) which, as it seems to us, are convenient for applications of the theorems of the preceding sections.

Theorem 7. In order that the zero solution of equation (2) be asymptotically stable, it is necessary and sufficient that there exist a continuously differentiable function $\varphi_1(t)$ $(t_0 \leq t \leq t_0 + \omega)$ such that

$$\varphi_1(t_0) \leq \varphi_1(t_0 + \omega), \qquad \int_0^\omega p(\tau)\, d\tau \geq 2 \int_{t_0}^{t_0+\omega} \varphi_1(\tau)\, d\tau \geq 0,$$

$$\varphi_1'(t) \leq \varphi_1^2(t) - p(t)\varphi_1(t) + q(t),$$

moreover, if

$$\int_{t_0}^{t_0+\omega} \varphi_1(\tau)\, d\tau = 0,$$

then

$$\dot{\varphi}_1(t^*) < \varphi_1^2(t^*) - p(t^*)\varphi_1(t^*) + q(t^*)$$

for some $t_0 \leqslant t^* \leqslant t_0 + \omega$.

We present one corollary of Theorem 7.

Theorem 8. Suppose that the function $q(t) \neq 0$. Suppose that the inequality

$$\int_0^{\omega} dt \int_t^{t+\omega} \exp\left[-\int_s^t p(\tau)\, d\tau\right] q(s)\, ds \geqslant 0$$

is satisfied. Then the zero solution of equation (2) is asymptotically stable.

Theorem 9. In order that the zero solution of equation (2) be unstable, it is necessary and sufficient that there exist a continuously differentiable function $\varphi_2(t)$ $(t_0 \leqslant t \leqslant t_0 + \omega)$ such that

$$\varphi_2(t_0) \geqslant \varphi_2(t_0 + \omega),$$

$$\int_{t_0}^{t_0+\omega} \varphi_2(\tau)\, d\tau \leqslant 0,$$

$$\dot{\varphi}_2(t) \geqslant \varphi_2^2(t) - p(t)\varphi_2(t) + q(t),$$

moreover, if

$$\int_{t_0}^{t_0+\omega} \varphi_2(\tau)\, d\tau = 0,$$

then

$$\dot{\varphi}_2(t^*) > \varphi_2^2(t^*) - p(t^*)\varphi_2(t^*) + q(t^*)$$

for some $t_0 \leqslant t^* \leqslant t_0 + \omega$.

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Note: Figure translations are in progress. See original paper for figures.