ON SOME INTEGRAL INEQUALITIES

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Abstract

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ON SOME INTEGRAL INEQUALITIES

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We generalize the well-known Chebyshev inequality for the case of one variable ([1]) to the case of n variables.

Theorem. Let P, Q, s be integrable functions of n variables $x_1, x_2, \dots, x_n$, defined on an open set Ω, where $s \geqslant 0$, and let the functions P and Q have continuous partial derivatives of the first order. Then, for the existence of the inequality

$$\int_\omega PQs\,dv \int_\omega s\,dv \geqslant \int_\omega Ps\,dv \int_\omega Qs\,dv, \tag{1}$$

where $dv = dx_1 dx_2 \dots dx_n$, in any domain $\omega \subset \Omega$, it is necessary and sufficient that the following conditions be satisfied:

a) P and Q must be functionally dependent, i.e. $P = f(Q)$;

b) f is a nondecreasing function of Q.

If f is a nonincreasing function, the sign of inequality (1) is reversed.

Proof of sufficiency. Form the difference

$$\Delta = \int_\omega PQs\,dv \int_\omega s\,dv - \int_\omega Qs\,dv \int_\omega Ps\,dv.$$

Changing the variables of integration and denoting the corresponding elementary volume by dw, we obtain

$$\Delta = \int_\omega \int_\omega s(x_1, \dots, x_n)\, s(y_1, \dots, y_n)\, P(y_1, \dots, y_n)[Q(y_1, \dots, y_n) - Q(x_1, \dots, x_n)]\,dv\,dw$$

or

$$\Delta = \int_\omega \int_\omega s(x_1, \dots, x_n)\, s(y_1, \dots, y_n)\, P(x_1, \dots, x_n)[Q(x_1, \dots, x_n) - Q(y_1, \dots, y_n)]\, dv\, dw.$$

Taking the half-sum of the obtained differences, we shall have

$$\Delta = \frac{1}{2} \int_\omega \int_\omega s(x_1, \dots, x_n)\, s(y_1, \dots, y_n)[P(x_1, \dots, x_n) - P(y_1, \dots, y_n)][Q(x_1, \dots, x_n) - $$

$$- Q(y_1, \dots, y_n)]\, dv\, dw. \qquad (2)$$

Substituting $P = f(Q)$, we obtain $\Delta \geqslant 0$. Thus the sufficiency is proved. We note that, for the proof of sufficiency, the requirement of the existence and continuity of the partial derivatives of the first order is superfluous.

Proof of necessity.

1. Let P and Q be functionally independent in Ω. We shall show that there exists a domain $\omega \subset \Omega$ for which inequality (1) is violated.

Indeed, in this case at least one of the second-order minors of the matrix

$$\begin{Vmatrix} \partial P/\partial x_1 & \dots & \partial P/\partial x_n \\ \partial Q/\partial x_1 & \dots & \partial Q/\partial x_n \end{Vmatrix}$$

does not vanish in Ω. Therefore there exists a point $r(x_1, \dots, x_n)$, in a neighborhood of which the vectors $\mathbf{P} = \operatorname{grad} P$ and $\mathbf{Q} = \operatorname{grad} Q$ are not collinear.

and the vector $\mathbf{R} = P((P+Q)Q) - Q((P+Q)P)$ is uniquely determined. Along the direction of the vector $\mathbf{R}$, in a sufficiently small neighborhood of the point $\mathbf{r}$, the integrand in (2) has a negative sign; consequently, inequality (1) is violated.

2. Let $P = f(Q)$, but let f be a nonmonotone function. We choose ω so that the values of Q within ω correspond to a region of decrease of the function f. Then in this region the integrand in (2) has a negative sign.

From inequality (1), under the assumption that $Q = P = p/q^2$, $s = p^2$, there follows Bunyakovsky’ s inequality

$$\left[\int_\omega pq\, dv\right]^2 \leqslant \int_\omega p^2\, dv \int_\omega q^2\, dv. \qquad (3)$$

Under certain conditions imposed on the functions P and Q, inequality (1) can be obtained from (3). For this purpose we rewrite (3) as follows:

$$\left[\int_\omega pq\,dv\right]^2 + \int_\omega \varphi\,dv \int_\omega \psi\,dv \leqslant \int_\omega p^2\,dv \int_\omega q^2\,dv + \int_\omega \varphi\,dv \int_\omega \psi\,dv,$$

where the functions φ, ψ are to be determined.

Next one may write that

$$\int_\omega \int_\omega [p(v)q(v)p(u)q(u) + \varphi(u)\psi(v)]\,du\,dv \leqslant \int_\omega \int_\omega [p^2(u)q^2(v) + \varphi(u)\psi(v)]\,du\,dv.$$

Thus, in order to pass from Bunyakovsky' s inequality to Chebyshev' s inequality, it suffices to put

$$p(v)q(v)p(u)q(u) + \varphi(u)\psi(v) = Q(u)s(u)P(v)s(v),$$

$$p^2(u)q^2(v) + \varphi(u)\psi(v) = P(u)Q(u)s(u)s(v). \tag{4}$$

One solution of the system of functional equations (4) is $\varphi = \lambda p^2$, $\psi = pq$. Hence

$$Qs = p(\lambda p + q), \qquad PQs = p^2, \qquad s = q(\lambda p + q), \qquad Ps = pq.$$

Another solution of the system of functional equations (4) is $\varphi = pq$, $\psi = \lambda q^2$. Hence

$$Ps = q(\lambda q + p), \qquad PQs = p(\lambda q + p), \qquad s = q^2, \qquad Qs = pq,$$

and, consequently,

$$Q = p/q, \qquad P = Q/(\lambda Q + 1) \tag{5}$$

or

$$Q = p/q, \qquad P = \lambda + Q. \tag{6}$$

It is easy to show that all other solutions of the system (4) lead to results analogous to (5) and (6).

Let in (1) $Q = f/s$ and $P = Q/(\lambda Q + 1)$. Then

$$\int_\omega \frac{f^2}{\lambda f + s}\, dv \int_\omega s\, dv \geqslant \int_\omega \frac{fs}{\lambda f + s}\, dv \int_\omega f\, dv;$$

for $\lambda = 0$

$$\int_\omega \frac{f^2}{s}\, dv \int_\omega s\, dv \geqslant \left[\int_\omega f\, dv\right]^2.$$

The last two inequalities occur in certain problems of mathematical statistics and information theory.

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CITED LITERATURE

1. G. M. Fikhtengol' ts, *Course of Differential and Integral Calculus*, 3, 1949.

Note: Figure translations are in progress. See original paper for figures.