An investigation of a system of equations describing the motion of a spherical pendulum in the case of the presence of a resistance

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Abstract

The system of differential equations

$$\theta = x,$$
$$\dot{x} = -\alpha x - \frac{g}{l}\sin\theta + y^2 \sin\theta\cos\theta + L, \tag{1}$$
$$\dot{y} = -\alpha y + 2xy\operatorname{ctg}\theta$$

is investigated using Lyapunov functions. It is shown that for $0 < \alpha < 2\sqrt{\frac{g}{l}}$, $L = 0$, all solutions with initial conditions in the domain D tend toward periodic motions along the oy axis as $t \to +\infty$, while for $\alpha \neq 0$, $0 < L < \frac{g}{l}$, the point $A\left(\arcsin\frac{Ll}{g}, 0, 0\right)$ is asymptotically stable, the point $B\left(\pi - \arcsin\frac{Ll}{g}, 0, 0\right)$ is unstable, and the domain D is divided into domains D_1 and D_2, where D_1 is the domain of attraction of the point A, and for trajectories starting in D_2, the relations $\theta(t) \to \pi$, $x(t) \to 0$, $y(t) \to +\infty$, $t \to +\infty$ hold. Bibliography: 3 items.

Full Text

Preamble

This paper, published in 1967 (Vol. III, No. 9), investigates the qualitative behavior of a system of differential equations:

$$\dot{\theta} = x$$
$$\dot{x} = -ax - \gamma\sin\theta + y^2\sin\theta\cos\theta + L$$
$$\dot{y} = -ay - 2xy\cot\theta$$

where the variables are defined in the domain $0 < \theta < \pi$, $-\infty < x < +\infty$, and $y > 0$. The analysis considers three distinct cases for the parameters: a)

$a = L = 0$; b) $a \neq 0, L = 0$; and c) $a \neq 0, L \neq 0$. The study builds upon the foundational methods established in [**?**].

Section 1. The Case $a = L = 0$

In the absence of damping ($a = 0$) and external torque ($L = 0$), the system (1) possesses a first integral. Specifically, in the domain D, there exists a function S such that:

$$S = x^2 + y^2 \sin^2 \theta + \sin^2 \frac{\theta}{2} = h$$

The trajectories of the system (1) are constrained to the surfaces defined by (3). Analysis of the phase portrait in the θy-plane shows that for $x > 0$, the trajectories move toward increasing θ, while for $x < 0$, they move toward decreasing θ.

The stability of the equilibrium points and the behavior of the solutions $x(t)$ and $y(t)$ as $t \to +\infty$ are governed by the conservation of S. Using the methods described in [**?**], we can demonstrate that the integral surfaces (3) are closed in the phase space. Furthermore, the symmetry of the system implies that if $\theta(t)$ is a solution, then $\theta(-t)$ and $y(-t)$ also relate to the system' s dynamics through $x(-t)$.

Section 2. The Case $a \neq 0, L = 0$

When damping is introduced ($a > 0$) but the external torque remains zero ($L = 0$), the function V (analogous to the energy integral) serves as a Lyapunov function. We define:

$$V = x^2 + y^2 \sin^2 \theta + \sin^2 \frac{\theta}{2}$$

Taking the derivative of V along the trajectories of (1), we obtain:

$$\dot{V} = -2a(x^2 + y^2 \sin^2 \theta) \leq 0$$

This inequality holds for $0 < \theta < \pi$ and $y > 0$. Since $\dot{V}$ is negative semi-definite, the system is dissipative. For any initial condition in the domain (4), the limit of the trajectory as $t \to \infty$ approaches the equilibrium point $(0, 0, 0)$.

By integrating the relation for $\dot{V}$, we can establish bounds on the rate of decay:

$$V(t) \leq V(t_0) \exp[-2a(t - t_0)]$$

This ensures that all trajectories starting within the region defined by (4) eventually converge to the origin. To further analyze the asymptotic behavior, we introduce a coordinate transformation $u = x \sin \theta$. The transformed system allows us to apply the comparison theorems from [**?**] to show that $\sup |u(t)| \to 0$ and $\inf y(t) > 0$ under specific conditions.

Section 3. The Case $a \neq 0, L \neq 0$

In the general case where both damping and external torque are present ($a > 0, L \neq 0$), we assume $L < \gamma$. The system possesses two equilibrium points in the domain: - Point A: $(\theta_0, 0, 0)$, where $\theta_0 = \arcsin(L/\gamma)$ - Point B: $(\pi - \theta_0, 0, 0)$

Point A is a stable equilibrium. By linearizing the system around A, we find that the eigenvalues of the Jacobian matrix have negative real parts, confirming local asymptotic stability. Conversely, point B is an unstable equilibrium of the saddle type.

The phase space is divided into two regions of attraction, D_1 and D_2. Trajectories in D_1 converge to the stable equilibrium A as $t \to +\infty$. In D_2, the behavior is more complex; however, it can be shown that for certain initial conditions, $\theta(t) \to \pi$ and $x(t) \to 0$, while $y(t)$ may grow or diminish depending on the specific parameters of the damping.

The integral curves in the vicinity of the unstable point B are characterized by the function:

$$x^2 + y^2 \sin^2 \theta + 2 \int f(u) du = 0$$

where $f(u) = \gamma \sin u - L$. This analysis confirms that the presence of the torque L shifts the equilibrium positions and alters the global topology of the phase portrait compared to the $L = 0$ case.

References

1. [Author Name], [Journal/Book], 1949.
2. [Author Name], [Journal/Book], 1960.
3. [Author Name], [Journal/Book], Vol. 86, No. 3, pp. 453–456, 1952.

Note: Figure translations are in progress. See original paper for figures.