ON CONDITIONS FOR THE APPLICABILITY OF ONE CLASS OF COMPUTATIONAL ALGORITHMS

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Abstract

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G. D. MAISTROVSKII

ON CONDITIONS FOR THE APPLICABILITY OF ONE CLASS OF COMPUTATIONAL ALGORITHMS

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Frequently used algorithms for the numerical solution of the functional equation

$$Ax = f \tag{1}$$

consist in a finite-dimensional approximation of the corresponding spaces and operator and the subsequent solution of the finite-dimensional equation obtained. S. L. Sobolev (^{1}) solved the problem of the applicability of such an algorithm in the case of a Fredholm integral equation. In the present work an attempt is made to generalize this result; namely, a class of algorithms is described that has, in a certain natural sense, a connection between the character of the finite-dimensional approximation and the method of solving the finite-dimensional equation. This latter circumstance makes it possible easily to establish necessary and sufficient conditions for the applicability of algorithms of this class in terms of the operator A.

The situation considered by us is explained by the diagram*

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "R^N"] \arrow[d, "P_z"'] & \mathfrak{M}^N \arrow[r, "A^N"] \arrow[d, shift right, "P_z"'] & \widetilde{\mathfrak{M}}^N \arrow[d, shift right, "\widetilde{P}_z"'] & \widetilde{\mathfrak{M}} \arrow[l, "\widetilde{R}^N"'] \arrow[d, "\widetilde{P}_z"'] \\
\mathfrak{M}_z & \mathfrak{M}_z^N \arrow[r, "A_z^N"] \arrow[u, shift right, "j_z^N"'] & \widetilde{\mathfrak{M}}_z^N \arrow[u, shift right, "\widetilde{j}_z^N"'] & \widetilde{\mathfrak{M}}_z
\end{tikzcd} \qquad (2)$$

Here $\mathfrak{M}$ is some linear normed space of real functions on the interval $[0, 1]$; the operator R^N $(N = 1, 2, ...)$ assigns to each $x \in \mathfrak{M}$ the piecewise-constant function $R^N x$:

$$(R^N x)(t) = R_k^N[x], \qquad t \in [t_{k-1}^N, t_k^N], \tag{3}$$

where $0 = t_0^N < t_1^N < \cdots < t_N^N = 1$ is a fixed (for the given N) partition of the interval $[0, 1]$; $\{R_k^N\}_{k=1}^N$ is a sequence of linear functionals on $\mathfrak{M}$, local in the sense that the values $R_k^N[x]$ depend only on the values of $x(t)$ on $[t_{k-1}^N, t_k^N)$. Suppose that $\mathfrak{M}^N \equiv \operatorname{Im} R^N \subset \mathfrak{M}$ and that the sequence $\{\mathfrak{M}^N\}$ approximates the space $\mathfrak{M}$ in the strong sense: $\lim_{N\to\infty} R^N x = x$ $(x \in \mathfrak{M})$.

The operators P_z $(0 < z \leq 1)$ appearing in (2) are defined by the equality

$$(P_z x)(t) = \begin{cases} x(t), & t \leq z, \\ 0, & t > z. \end{cases} \tag{4}$$

Suppose that $\mathfrak{M}_z = \operatorname{Im} P_z \subset \mathfrak{M}$. Let $\mathfrak{M}_z^N \equiv \operatorname{Im} P_z R^N$. Then there exists, obviously, a natural mapping $j_z^N : \mathfrak{M}_z^N \to \mathfrak{M}^N$, defined by the formula

$$j_z^N(P_z R^N x) = P_{t_k} R^N x \qquad (k = \min\{j :\ t_j \geq z\}). \tag{5}$$

We have described the left-hand part of diagram (2). The right-hand part is constructed analogously for some, generally speaking, different functional space $\widetilde{\mathfrak{M}}$, with the partitions $\{t_k^N\}$ preserved.

Let $\mathfrak{S} \subset \mathfrak{M}$, $\widetilde{\mathfrak{S}} \subset \widetilde{\mathfrak{M}}$ be certain subspaces,* and let $A : \mathfrak{S} \to \widetilde{\mathfrak{S}}$ and $A^N : \mathfrak{M}^N \to \widetilde{\mathfrak{M}}^N$ $(N = 1, 2, \ldots)$ be certain linear operators, where the sequence of operators $A^N R^N$ converges strongly to A on $\mathfrak{S}$. Put

$$A_z^N \equiv \widetilde{P}_z A^N j_z^N.$$

Let, further, $\mathfrak{S}'_z \equiv \operatorname{Im} P_z|_{\mathfrak{S}}$; $\mathfrak{S}_z$ is the subspace in $\mathfrak{S}'_z$ on which the sequence of operators $\widetilde{P}_z A^N R^N$ converges strongly; A_z is the limiting operator. We assume that $\widetilde{\mathfrak{S}}_z \equiv \operatorname{Im} A_z \subset \widetilde{\mathfrak{S}}'_z$. The situation described is represented in the diagram

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "P_z"] \arrow[d, "A"] & \mathfrak{S}'_z & \mathfrak{S}_z \arrow[l, "i_z"'] \arrow[d, "A_z"] \\ \widetilde{\mathfrak{S}} \arrow[r, "\widetilde{P}_z"] & \widetilde{\mathfrak{S}}'_z & \widetilde{\mathfrak{S}}_z \arrow[l, "\tilde{i}_z"'] \end{tikzcd} \tag{6}$$

where i_z and $\tilde{i}_z$ are embedding operators. Note that if $x \in \mathfrak{S} \cap \mathfrak{S}_z$, then $A_z x = \widetilde{P}_z A x$.

The algorithms under consideration for the numerical solution of equation (1) in the space $\mathfrak{S}$, with right-hand side $f \in \widetilde{\mathfrak{S}}$, consist in approximating equation (1) by the finite-dimensional equation

$$A^N y = \widetilde{R}^N f \tag{1'}$$

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* This diagram is, generally speaking, noncommutative.

in the space $\mathfrak{M}^N$, with right-hand side from $\widetilde{\mathfrak{M}}^N$, and solving (1) by some method based on triangular factorization of the matrix (without preliminary rearrangement). In doing so, it is assumed that the following **basic condition** of correspondence of the method of approximation to the class of operators under consideration (b.c.) is fulfilled: the mappings $A_z : \mathfrak{S}_z \to \widetilde{\mathfrak{S}}_z$ $(0 < z \leq 1)$ are one-to-one if and only if, for all sufficiently large N, the mappings $A_z^N : \mathfrak{M}_z^N \to \widetilde{\mathfrak{M}}_z^N$ are one-to-one, and moreover

$$(A_z^N)^{-1}\widetilde{R}^N \xrightarrow[N\to\infty]{} A_z^{-1} \tag{7}$$

on the space $\widetilde{\mathfrak{S}}_z$ in the sense of strong convergence.

We shall denote the described class of algorithms by the symbol H.

Definition. We shall say that an algorithm of class H is applicable to equation (1) if, for all sufficiently large N, the system (1) is solvable by this algorithm, and the solution of system (1) converges to the solution of equation (1) in the norm of the space $\widetilde{\mathfrak{M}}$.

Theorem 1. In order that algorithms of class H be applicable to equation (1), it is necessary and sufficient that all mappings $A_z : \mathfrak{S}_z \to \widetilde{\mathfrak{S}}_z$ $(0 < z \leq 1)$ be one-to-one.

Indeed, in the coordinate system induced in the spaces $\mathfrak{M}_z^N$ and $\widetilde{\mathfrak{M}}_z^N$, the matrix of the operator A_z^N is the matrix composed of the first k rows and the first k columns of the matrix of the operator A^N, where $k(z) = \min\{j : t_j \geq z\}$. Hence it follows that the condition for solvability of system (1) by algorithms of class H, i.e. the nonvanishing of all principal minors of the matrix of this system (2), is equivalent to the nonsingularity of the operators A_z^N, which, by virtue of the b.c., is equivalent to the existence of the operators A_z^{-1} on the space $\widetilde{\mathfrak{S}}_z$.

The convergence of the solution of system (1) to the solution of equation (1) is ensured by the limiting relation (7).

We give two examples of application of this theorem.

Example 1. Numerical solution of a boundary-value problem for an ordinary linear differential operator*. Let $\mathfrak{M} = \widetilde{\mathfrak{M}} = D[0,1]$ be the space of piecewise-continuous functions on $[0,1]$ with the uniform metric; $\mathfrak{S}$ is the subspace of all twice continuously differentiable functions on $[0,1]$ satisfying the boundary conditions

$$y'(0) - H_0 y(0) = 0, \qquad y'(1) + H_1 y(1) = 0, \tag{8}$$

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* Closedness of the subspaces is not assumed.

$\widetilde{\mathfrak{S}} = C[0,1]$, and A is a differential operator of the form

$$Ay = ay'' + by' + cy,$$

where $a(x), b(x), c(x) \in C[0,1]$ and $a(x) > 0$ on $[0,1]$. It is assumed that the boundary-value problem generated by the operator A is nondegenerate (the point $\lambda = 0$ is not an eigenvalue of the operator A).

We perform the usual mesh approximation of the spaces $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ and of the operator A (with constant step $h > 0$; in this case the last interval may turn out to be smaller than h). Let $N = [1/h]$. Then $\mathfrak{S}_z$ is the set of all twice continuously differentiable functions on $[0, z]$ satisfying the boundary conditions

$$y'(0) - H_0 y(0) = 0, \qquad y(z) = 0; \tag{9}$$

A_z is the differential operator

$$A_z y = ay'' + by' + cy,$$

defined on $\mathfrak{S}_z$. We note that, for the differential operator under the approximation method under consideration, the b.c. is satisfied.

We shall solve system $(1')$ by the sweep method, or by the method of square roots (if the operator A is self-adjoint), or by the compact Gaussian scheme. Together with the approximation method described, these methods constitute certain algorithms of class H. Theorem 1 leads to the following result.

Theorem 2. *In order that algorithms of class H be applicable to the nondegenerate boundary-value problem*

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

$$y'(0) - H_0 y(0) = 0, \qquad y'(1) + H_1 y(1) = 0$$

it is necessary and sufficient that the solution of the Cauchy problem

$$ay'' + by' + cy = 0,$$

$$y(0) = 1, \qquad y'(0) = H_0$$

not vanish on the interval $[0,1]$.

We note that if A is a self-adjoint operator, this condition is equivalent to its positive definiteness in $L^2[0,1]$. This case was considered in work (3).

Example 2. Numerical solution of a linear integral equation of Fredholm type of the second kind with a discontinuous kernel. Let $\mathfrak{M} = \widetilde{\mathfrak{M}} = D[0,1]$, $\mathfrak{S} = \widetilde{\mathfrak{S}} = C[0,1]$. The algorithm under consideration consists of a mesh approximation of equation (1) and the solution of the resulting system by one of the methods based on triangular factorization of the matrix. We assume that the b.c. is satisfied. This is the case, for example, for the algorithm T_c, in which the approximation satisfies the requirements of a proper approximation (1). Consequently, from Theorem 1 it follows

Theorem 3 (S. L. Sobolev [1]). The algorithm T_c is applicable if and only if each of the integral equations of the family ($z \in [0,1]$—a parameter)

$$\varphi(x) = \int_0^z K(x,y)\varphi(y)\,dy$$

has only the trivial solution.

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Kharkov Institute of Mining Machinery Automation and Computer Engineering

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Note: Figure translations are in progress. See original paper for figures.

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[*] For simplicity we shall consider an operator of the second order; however, the result is carried over directly to an arbitrary linear differential operator.