A GENERATING FUNCTION FOR CASIMIR OPERATORS

V. S. POPOV, A. M. PERELOMOV

1967

SovietRxiv

Abstract

Full Text

UDC 513.83

MATHEMATICS

V. S. POPOV, A. M. PERELOMOV

A GENERATING FUNCTION FOR CASIMIR OPERATORS

(Presented by Academician I. G. Petrovskii, 11 VIII 1966)

Invariant operators constructed from the generators of a group (the so-called Casimir operators) have been studied in many works (see, for example, ([1],[2]) and the further references in ([3],[4])). Their importance for group theory is determined by the well-known fact that the eigenvalues of the Casimir operators uniquely specify an irreducible representation of the group. The authors ([3],[4]) obtained explicit formulas for the eigenvalues of the Casimir operators in the case of all classical groups. In the present note we give formulas for the generating function $G(z)$ of the Casimir operators

$$G(z) = \sum_{p=0}^{\infty} C_p(\mathbf{m}) z^p, \tag{1}$$

knowledge of which substantially simplifies the computation of the eigenvalues $C_p(\mathbf{m})$.

If in formulas (8), (15) of ([4]) the matrix a_{ij} is put in diagonal form, then we obtain the following expression* for $C_p(\mathbf{m})$:

$$C_p(\mathbf{m}) = \sum_{(i)} \xi_i \Lambda_i^p, \tag{2}$$

$$\xi_i = \begin{cases} \prod\limits_{j \neq i} \left(1 - \dfrac{1}{l_i - l_j}\right), & \text{for the group } A_n; \\ \left(1 - \dfrac{1-\beta}{2l_i}\right) \prod\limits_{j \neq \pm i} \left(1 - \dfrac{1}{l_i - l_j}\right), & \text{for the groups } B_n,\ C_n,\ D_n,\ \text{if } i \neq 0; \\ 1, & \text{if } i = 0. \end{cases}$$

Here $\Lambda_i = l_i + \alpha + \frac{1}{2}\delta_{i0}$ are the eigenvalues of the matrix a_{ij}; for $i \neq 0$, $\Lambda_i = l_i + \alpha \equiv \lambda_i$ (the equality $i = 0$ can occur only in the case of the group $O(2n+1)$, see the table in ([4])). Formula (2) can be represented in the form of a contour integral

$$C_p(\mathbf{m}) = \frac{1}{2\pi i} \oint_{(+)} \left[\frac{\lambda - (\alpha + (1-\beta)/2)}{\lambda - (\alpha + 1/2)} (1 - f(\lambda)) + 1 - \beta \right] \lambda^p \, d\lambda, \qquad (3)$$

where

$$f(\lambda) = \prod_{(i)} \left(1 - \frac{1}{\lambda - \lambda_i} \right) \quad (\lambda_i = l_i + \alpha), \qquad (4)$$

and the contour of integration encloses all the poles of the integrand. Hence follows the following expression** for the generating

* I. M. Gelfand drew our attention to the important significance of formula (2) for $C_p(\mathbf{m})$.

** It should be noted that the derivation given above of formula (5) for $G(z)$ does not cover several special cases. Such exceptional cases are: 1) representations of the group $O(2n+1)$ for which $m_n = 0$; 2) representations of the group $O(2n)$ with $m_n = \pm\frac{1}{2}$. For example, in case (1) the matrix a_{ij} has two equal eigenvalues and is not reducible to diagonal form. It can be shown, however, that the final formulas (5), (6) for the generating function $G(z)$ retain their form for the exceptional cases listed.

functions of the Casimir operators:

$$G(z) = \frac{1 - (\alpha + (1-\beta)/2)z}{1 - (\alpha + 1/2)z} \, \frac{1 - \Pi(z)}{z}, \qquad (5)$$

where

$$\Pi(z) = \prod_{(i)} \left(1 - \frac{z}{1 - \lambda_i z} \right). \qquad (6)$$

Formulas (5), (6) hold for all classical groups*. The values of the parameters α, β characterize the group uniquely and can be found from the tables in papers ([3,4]).

From formulas (5), (6) there follows in a trivial way the well-known Racah theorem, according to which the eigenvalues $C_p(\mathbf{m})$, expressed in terms of the components of the vector $\mathbf{l} = \mathbf{m} + \mathbf{r}$ ($\mathbf{m}$ is the highest weight of an irreducible representation, $\mathbf{r}$ is the half-sum of the positive roots of the algebra), possess the symmetries S of the Weyl group (the group of reflections in hyperplanes orthogonal to the root vectors). The transformations that constitute the Weyl group are expressed especially simply in the tensor basis (i.e., in the language of the variables $l_i = m_i + r_i$): for the group A_{n+1} they consist of all permutations of the numbers $l_1, l_2, \dots, l_n$ among themselves; for the groups B_n, C_n, D_n, in addition to these permutations, "inversions" are included: $l_i \to l_{-i} = -l_i$, $l_j \to l_j$ $(j \neq i)$ (for the proof see ([2])). It is immediately evident that the function $\Pi(z)$ does not change its form under the performance of the transformations described above on the numbers l_i, which proves Racah' s theorem. The proofs of this theorem known in the literature ([2],[5]) are more complicated.

Let us give a number of simple examples of the use of formula (5). For the identity representation $(m_i = 0)$ we have:

$$G_0(z) = 2\alpha + 1 + \beta = d, \qquad C_p(0) = \begin{cases} d & \text{for } p = 0, \\ 0 & \text{for } p > 0 \end{cases} \tag{7}$$

(here d is the dimension of the matrix a_{ij}). For spinor representations in the rotation groups $\beta = 1$, $\mathbf{m} = (1/2, \dots, 1/2, \pm 1/2)$, whence

$$G(z) = \frac{2(\alpha+1)(1-\alpha z)}{(1+z/2)(1-(\alpha+1/2)z)}, \qquad C_p = (\alpha{+}1/2)\left[(\alpha+1/2)^{p-1} - (-1/2)^{p-1}\right]. \tag{8}$$

The simplest representations of the classical groups are the completely symmetric or antisymmetric ones. The eigenvalues C_p for these representations also follow from (5); explicit formulas can be found in ([3]). We shall consider here a more general case, but restrict ourselves only to the unitary group.

Let the Young diagram have the form of a rectangle of f rows and k columns; then for the group $U(n)$ we obtain:

$$G(z) = n + \frac{fkz}{1-(n+f-k)z}, \qquad C_p(\underbrace{f, \dots, f}_{k \text{ times}}, 0, \dots, 0) = fk(n+f-k)^{p-1}. \tag{9}$$

Finally, let us note that concrete computations of $C_p(\mathbf{m})$ are conveniently performed by expanding $C_p(\mathbf{m})$ in the standard power sums S_k:

$$S_k = \sum_{(i)} (l_i^k - r_i^k) = \varepsilon \sum_{i=1}^{n} (l_i^k - r_i^k), \tag{10}$$

where

$$\varepsilon = \begin{cases} 1 & \text{for the group } A_n, \\ 1+(-1)^k & \text{for the groups } B_n,\ C_n,\ D_n \end{cases} \tag{11}$$

A particularly simple form of $G(z)$ is obtained for unitary groups:

$$\beta = 0, \qquad G(z) = \frac{1-\Pi(z)}{z}.$$

(for the identity representation all $\tilde{S}_k$ vanish). The corresponding expansion can be obtained from (5), (6), if in (6) one passes to logarithms. It has the form

$$G(z) = de^{-\psi(z,\alpha)} + \frac{1-\left(\alpha+\frac{(1-\beta)}{2}\right)z}{1-(\alpha+1/2)z}\,\frac{1-e^{-\psi(z,\alpha)}}{z}. \tag{12}$$

Here

$$d = 2\alpha + 1 + \beta,$$

$$\psi(z,\alpha) = \sum_{k=2}^{\infty} \tilde{a}_k(\alpha) z^k, \qquad \tilde{a}_k(\alpha) = \sum_{l=1}^{k-1} \frac{(k-1)!}{l!(k-l)!}\left[(\alpha+1)^{k-l} - \alpha^{k-l}\right]\tilde{S}_l. \tag{13}$$

Introducing the quantities $B_p(\alpha)$, defined by the expansion

$$e^{-\psi(z,\alpha)} = -\sum_{p=0}^{\infty} B_{p-1}(\alpha) z^p, \qquad B_{-1} = -1, \tag{14}$$

we find for the eigenvalues of the Casimir operators the formula

$$C_p(\mathbf{m}) = B_p - dB_{p-1} + \frac{\beta}{2}\sum_{r=1}^{p-1}(\alpha+1/2)^{p-r-1}B_r. \tag{15}$$

For the unitary group $\beta = 0$, and (15) goes over into formula (14) of ([4]). For the groups B_n, C_n, and D_n one must take into account that all $\tilde{S}_k$ with odd k vanish, which substantially simplifies formulas (13)–(15). As a result we obtain expressions for the simplest Casimir operators on the groups B_n, C_n, D_n:

$$C_1 = 0, \qquad C_2 = \tilde{S}_2, \qquad C_3 = (\alpha + (1-\beta)/2)\tilde{S}_2,$$

$$C_4 = \tilde{S}_4 - (\alpha\beta + (\beta - 1)/2)\tilde{S}_2, \tag{16}$$

found earlier in ([3]) by a more complicated method.

In conclusion the authors express their sincere gratitude to I. M. Gelfand for his interest in the work and for discussion of the results.

Received
30 VI 1966

REFERENCES

[1] I. M. Gelfand, *Matem. sborn.*, **26**, 103 (1950).
[2] G. Racah, *Group Theory and Spectroscopy*, Lecture Notes, Princeton, 1951.
[3] A. M. Perelomov, V. S. Popov, *Pis' ma v redaktsiyu ZhETF*, **1**, no. 6, 15; **2**, no. 1, 34 (1965); *Yadernaya fizika*, **3**, 924, 1127 (1966).
[4] A. M. Perelomov, V. S. Popov, *DAN*, **174**, No. 2 (1967).
[5] B. Gruber, L. O' Raifeartaigh, *J. Math. Phys.*, **5**, 1796 (1964).

Note: Figure translations are in progress. See original paper for figures.