ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR A HIGHER-ORDER ELLIPTIC EQUATION IN THE SPACES \(C_{l+\alpha(r)}\)

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Abstract

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ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR A HIGHER-ORDER ELLIPTIC EQUATION IN THE SPACES $C_{l+\alpha(r)}$

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In [1] sharp estimates were obtained for the derivatives of solutions of the Dirichlet problem for a second-order elliptic equation, under the condition that the coefficients and the right-hand side of the equation, as well as the second derivatives of the boundary function, had a modulus of continuity not exceeding a certain function $\omega(r)$ satisfying Dini' s condition. It was proved that the second derivatives of the solution have a modulus of continuity not exceeding

$$C\int_0^r \frac{\omega(t)}{t}\,dt.$$

On fundamental solutions see [4, 5].

Theorems 1 and 4 contain sharp interior estimates for solutions of elliptic and parabolic systems.

In Theorem 2, estimates are obtained for solutions of general boundary-value problems in a half-ball, on the flat part of whose boundary general boundary conditions are prescribed. With the aid of Theorems 1 and 2 we shall obtain estimates for solutions of general boundary-value problems for an arbitrary domain with sufficiently smooth boundary. In Hölder norms these estimates were obtained in [2, 3].

Let us introduce the following definitions: we shall say that a function $f(x)$ satisfies a uniform Hölder condition in the domain D with refined exponent $\alpha(r)$ if, for any points $x, x' \in D$,

$$|f(x) - f(x')| \le C|x - x'|^{\alpha(|x-x'|)};$$

the function $f(x)$ satisfies a local Hölder condition with refined exponent $\alpha(r)$ if it satisfies a uniform Hölder condition with refined exponent $\alpha(r)$ in every subdomain D' of the domain D that is strictly interior with respect to D, where the function $\alpha(r)$ is defined and continuous for $0 < r < \infty$ and satisfies the following conditions: $\alpha(r) \to \lambda$, $\lambda \in [0,1)$, $\alpha'(r) r \ln r \to 0$ both as $r \to 0$ and as $r \to \infty$, and, if $\lambda = 0$, then $\alpha(r)\ln r \to -\infty$ as $r \to 0$, and $\alpha(r) + r \ln r \cdot \alpha'(r) > 0$ for $r \in R_0 = (0, r_0) + (1/r_0, \infty)$, where r_0 is some sufficiently small number. (It is assumed that $\alpha'(r)$ exists and is continuous, at least for sufficiently small and sufficiently large r.) From these conditions it follows that the function $r^{\alpha(r)}$ increases monotonically for $r \in R_0$.

The function $r^{\alpha(r)-\lambda}$ is slowly varying, i.e.

$$(kr)^{\alpha(kr)-\lambda} / r^{\alpha(r)-\lambda} \to 1$$

as $r \to 0$ and as $r \to \infty$, uniformly with respect to k, $0 < a \le k \le b < \infty$.

We denote by $C^{\alpha(r)}(D)$ the class of functions satisfying a local Hölder condition with refined exponent $\alpha(r)$ in the domain D. By $C^{l+\alpha(r)}(D)$ we denote the class of functions for which $D^l f(x) \in C^{\alpha(r)}(D)$.

By $C_{l+\alpha(r)}(D)$ we denote the space of functions for which the norm is finite:

$$|f|^D_{l+\alpha(r)} = \sum_{k=0}^{l} \sup_{\substack{x \in D \\ |j|=k}} |D^j f(x)| + \sup_{\substack{x,x' \in D \\ |j|=l}} \frac{|D^j f(x) - D^j f(x')|}{|x-x'|^{\alpha(|x-x'|)}} \equiv \sum_{k=0}^{l} [f]_k + [f]_{l+\alpha(r)}.$$

By $C_{p+\beta(r),\, l+\alpha(r)}(D)$ we mean the space of functions for which the norm

$$|f|^D_{p+\beta(r),\, l+\alpha(r)} = \sum_{k=0}^{l} \sup_{\substack{x \in D \\ |j|=k}} d_x^{p+k} |D^j f(x)| +$$

$$+ \sup_{\substack{x,x' \in D \\ |j|=l}} d_{xx'}^{p+l+\beta(d_{xx'})+\alpha(d_{xx'})} \frac{|D^j f(x) - D^j f(x')|}{|x-x'|^{\alpha(|x-x'|)}} \equiv \sum_{k=0}^{l} [f]_{p,k} + [f]_{p+\beta(r),\, l+\alpha(r)},$$

is finite, where d_x is the distance from the point x to the boundary of the domain D and $d_{xx'} = \min(d_x, d_{x'})$, $p + l \ge 0$, p is an integer, $\beta(r)$ is some function, $\alpha(r) + \beta(r) \ge 0$.

By $\mathcal{A}_l$ we denote the class of refined exponents $\alpha(r)$ for which $A^l \alpha(r) < \infty$, where

$$A\alpha(r) = \int_0^r r^{\alpha(r)-1}\, dr, \qquad l = 1, 2, \ldots$$

Let $\alpha(r) \in \mathcal{A}_1$ and let

$$B\alpha(r) = \frac{1}{\ln r} \ln \frac{A\alpha(r)}{A\alpha(1)}.$$

Theorem 1. Let the functions $u_j(x) \in C_j^t(D)$ satisfy the elliptic system in the sense of Douglis–Nirenberg

$$\sum_{j=1}^{N} l_{ij}(x, D) u_j(x) = f_i(x), \qquad i = 1, 2, \dots, N, \tag{1}$$

$$l_{ij}(x, D) = \sum_{|p| \le s_i + t_j} a_{ij,p}(x) D^p, \qquad s_i,\ t_i \text{ are integers}, \quad \max s_i = 0.$$

Suppose that

$$|a_{ij,p}|_{-\alpha(r),\,-s_i+l+\alpha(r)}, \qquad |p| = s_i + t_j,$$

$$|a_{ij,p}|_{s_i+t_j-|p|+B\alpha(r)-\alpha(r),\,-s_i+l+\alpha(r)}, \qquad |p| < s_i + t_j,$$

are bounded by the constant K_1; $\alpha(r) \in \mathcal{A}_1$, $l \ge 0$ is an integer. Then

$$|u_j|_{t-t_j,\,t_j+l+B\alpha(r)} \le$$

$$\le K_2 \left(\sup_i |f_i|_{s_i+t+B\alpha(r)-\alpha(r),\,-s_i+l+\alpha(r)} + \sum_{t_j>0} |u_j|_{t-t_j,\,0} \right), \qquad t = \max_i t_i, \tag{2}$$

where K_2 depends on K_1, n, the ellipticity constant and on the function $A\alpha(r)$, the domain D, s_i and t_i.

In the proof the following lemma is used.

Lemma 1. Let

$$u(x) = \int_S D_y^{2m} \Gamma(x-y, a(x))(g(y) - g(x)) h(y)\, dy,$$

where $S = \{|x_0 - x| \le d,\ d \le 1\}$, $\Gamma(x, a(\xi))$ is the fundamental solution of the elliptic equation with constant coefficients

$$\sum_{|k|=2m} a_k(\xi) D_x^k u(x) = 0.$$

Let $g(x) \in C_{\alpha(r)}(S)$, $h(x) \in C_{\gamma(r)}(S)$, where $\alpha(r) \in \mathcal{A}_1$,

$$\gamma(r) = \frac{1}{\ln r} \ln \left(r^{B\alpha(r)} + r^{\alpha(r)} \ln \frac{1}{r} \right)$$

and let

$$[a]^S_{\alpha(r)} = \sum_{|k|=2m} [a_k]^S_{\alpha(r)} < \infty.$$

Then the estimate holds

$$|u(x_0) - u(x)| \leq C[g]^S_{\alpha(r)}(1 + [a]^S_{\alpha(r)})|h|^S_{\gamma(r)}|x_0 - x|^{B\alpha(|x_0 - x|)}$$

where the constant C depends on n, $A\alpha(r)$, and the ellipticity constant.

Let now Σ_R be a half-ball in the space E_{n+1}, $\Sigma_R = \{|x| < R,\ x_{n+1} \geqslant 0\}$, $R \leqslant 1$. Consider in Σ_R the problem

$$L(x, D)u(x) = f(x) \tag{3}$$

$$B_j(x', D)u(x)\big|_{x_{n+1}=0} = \varphi_j(x), \qquad x' = (x_1, x_2, \dots, x_n), \qquad j = 1, \dots, m,$$

and suppose: 1) the operators L and B_j satisfy conditions i) and ii) of [3], p. 70 (the leading coefficients of L do not depend on x_{n+1} if $\lambda = 0$); 2) the coefficients of the operators L and B_j are bounded by the constant K_3 in the norms $|\cdot|_{l-2m+\alpha(r)}$ and $|\cdot|_{l-m_j+B\alpha(r)}$, $l \geqslant l_0 = \max(2m, m_j)$, respectively, $\alpha(r) \in A_4$.

Theorem 2. Let the function $u(x) \in C^{l_0+B^2\alpha(r)}(\Sigma_R)$ be a solution of problem (3). Let assumptions 1), 2) be fulfilled. Suppose that the norms

$$|\bar{f}|_{2m+B^2\alpha(r)-\alpha(r),\, l-2m+\alpha(r)}, \qquad |\varphi_j|_{m_j+B^2\alpha(r)-B\alpha(r),\, l-m_j+B\alpha(r)}$$

and $|u|_0$ are finite, $\alpha(r) \in A_4$. Then the estimate holds

$$|\bar{u}|_{0,\, l+B^2\alpha(r)} \leqslant K_4 \left(|\bar{f}|_{2m+B^2\alpha(r)-\alpha(r),\, l-2m+\alpha(r)} + \sum_{j=1}^{m} |\varphi_j|_{m_j+B^2\alpha(r)-B\alpha(r),\, l-m_j+B\alpha(r)} + |u|_0 \right), \tag{4}$$

where K_4 depends on n, on the function $A^4\alpha(r)$, K_3, and on the constants entering condition 1). (The bar over the norm means that the distance entering its definition is measured only up to the curvilinear boundary Σ_R.)

Estimate (4) is proved from a certain representation for the derivatives $D^l u(x)$ and with the aid of Lemma 2.

Lemma 2. Let

$$u(x) = \int K(x' - y', x_{n+1}) f(y')\, dy', \qquad x_{n+1} > 0,$$

where

$$K(x) = W(x/|x|)|x|^{-n},$$

$W(Q)$ is a continuously differentiable function on the half-sphere $|Q| = 1$, $x_{n+1} \geqslant 0$, and

$$\int_{|\xi|=1} W(\xi', 0)\, d\Omega_\xi = 0.$$

Let $f(x') \in C_{\alpha(r)}(E_n)$, $\alpha(r) \in A_1$, and let $f(x') \in L_p$ for some finite $p \geqslant 1$. Then

$$[u]_{B\alpha(r)} \leqslant C[f]_{\alpha(r)},$$

where C depends on n, $\max|W|$, $\max|DW|$, and $A\alpha(r)$.

Lemma 3. Let

$$u(x) = \int K(x' - y', x_{n+1}) \int D^{2m}\Gamma(y' - \eta', -\eta_{n+1})(g(\eta) - g(x_0', 0))h(\eta)\, d\eta\, dy',$$

$$x_{n+1} > 0,$$

where the kernel $K(x)$ satisfies the conditions of Lemma 2, while $g(x) \in C_{\alpha(r)}(E_{n+1})$, $h(x) \in C_{B^2\alpha(r)}(E_{n+1})$ and is finite, and the function $\Gamma(x)$ is the fundamental solution of an elliptic equation with constant coefficients

$$\sum_{|k|=2m} a_k D^k u(x) = 0, \qquad \alpha(r) \in A_4.$$

Then

$$|u(x_0) - u(x)| \leqslant C[g]_{\alpha(r)} \left(|h|_0 + |x_0 - x|^{B^4\alpha(|x_0 - x|)}[h]_{B^2\alpha(r)}\right) |x_0 - x|^{B^2\alpha(|x_0 - x|)},$$

where the constant C depends on n, $\max|W|$, $\max|DW|$, and on the function $A^4\alpha(r)$ and the constant of ellipticity.

Let now D be an arbitrary domain, which may be unbounded, in the space E_{n+1}. We shall denote its boundary by ∂D. Consider in the domain D the problem

$$Lu(x) = f(x), \qquad B_j u(x)\big|_\Gamma = \varphi_j(x), \qquad j = 1, \dots, m, \tag{5}$$

where Γ is a part of the boundary ∂D, which may coincide with the entire boundary of the domain D; L and B_j are differential operators of orders $2m$ and m_j. Put $l_0 = \max(2m, m_j)$ and let $l \geqslant l_0$ be some integer and $\alpha(r) \in A_4$.

Consider a subdomain (possibly unbounded) U of the domain D, such that $\partial U \cap \partial D \subset \Gamma$. We shall assume that $\Gamma \in C^{l+\alpha(r)}$. Suppose that for U and D the conditions of [3], p. 76, are fulfilled, and it is assumed that each component of the mapping T_P (see [3], p. 76) and its inverse has norm $|\cdot|_{l+\alpha(r)}$ bounded by the constant K_5, independent of P. With respect to the equation and the boundary conditions we shall assume that, under any such mapping T_P, they pass into system (3) in the half-ball $\Sigma_{R(P)}$, satisfying conditions 1), 2). -

assume that the coefficients of the operators L and B_j have, respectively in the domain D, finite K_6-norms $|\cdot|_{l-2m+\alpha(r)}$ and $|\cdot|_{l-m_j+B\alpha(r)}$, and that the operator L is uniformly elliptic in this domain.

Let $|f|^D_{l-2m+\alpha(r)}$, $|\varphi_j|^\Gamma_{l-m_j+B\alpha(r)}$, and $|u|^D_0$ be finite, $\alpha(r) \in \mathcal{A}_4$.

Theorem 3. Let the function $u(x) \in C^{l+B^2\alpha(r)}$ in $D+\Gamma$ be a solution of problem (5). Then

$$u(x) \in C^{l+B^2\alpha(r)} \quad \text{in } \overline{U},$$

$$|u|^U_{l+B^2\alpha(r)} \leq K_7 \left(|f|^D_{l-2m+\alpha(r)} + \sum |\varphi_j|^\Gamma_{l-m_j+B\alpha(r)} + |u|^D_0 \right),$$

where the constant K_7 depends on d, K_5, m, m_j, K_6, on the function $A^4\alpha(r)$, and on the constants entering condition 1).

If the domain D is bounded, then in Theorem 3 one may take $U = D$ and $\Gamma = \partial D$. In this case the normal solvability of problem (5) follows from Theorem 3.

For $\alpha(r) \in \mathcal{A}_1$, interior estimates have been obtained for parabolic systems in the sense of Petrovsky (for second order, see ([6])). For brevity we formulate the result for a single equation. Let D be a bounded domain in $(n+1)$-dimensional space $(x_1, x_2, \dots, x_n, t)$. Introduce the distance

$$d(P,Q) = \sqrt{|x-\xi|^2 + |t-t'|^{1/m}}. \qquad (*)$$

Let

$$|f|_{p+3(r),\alpha(r)} = \sup_{P\in D} d_P^p |f(P)| + \sup_{P,Q\in D} d_{PQ}^{p+\alpha(d_{PQ})+\ell(d_{PQ})} \frac{|f(P)-f(Q)|}{d(P,Q)^{\alpha(d(P,Q))}},$$

where $p \geq 0$ is an integer, d_P is the distance in the sense of $(*)$ from the point $P(x,t)$ to the boundary of D lying in the half-space $t < \tau$; $d_{PQ} = \min(d_P, d_Q)$.

Theorem 4. Let $u(x,t)$ have continuous derivatives with respect to x up to order $2m$ and a continuous derivative with respect to t, and satisfy the parabolic equation

$$\sum_{|k|\leq 2m} a_k(x,t) D_x^k u(x,t) - D_{tu}(x,t) = f(x,t).$$

Suppose that $|a_k|_{-\alpha(r),\alpha(r)}$, $|k| = 2m$, and $|a_k|_{2m-|k|+B\alpha(r)-\alpha(r),\alpha(r)}$, $|k| < 2m$, are bounded by the constant $\widetilde{K}$. Suppose that $|f|_{2m+B\alpha(r)-\alpha(r),\alpha(r)} < \infty$, $\alpha(r) \in \mathcal{A}_1$. Then

$$|u|_{0,\,2m+B\alpha(r)} \leq K \left(|f|_{2m+B\alpha(r)-\alpha(r),\alpha(r)} + |u|_0 \right),$$

where

$$|u|_{0,\,2m+B\alpha(r)} = \sum_{|k|<2m} \sup_{P\in D} d_P^{|k|} \left| D_x^k u(P) \right| + \sum_{|k|=2m} |D_x^k u|_{2m,B\alpha(r)} + |D_{tu}|_{2m,B\alpha(r)}$$

and the constant K depends on the constant of parabolicity, $\widetilde{K}$, n, the diameter of the domain D, and the function $\alpha(r)$.

For $\alpha(r) \in \mathcal{A}_2$, Schauder-type estimates have been obtained for solutions of the first boundary-value problem for parabolic systems in the sense of Petrovsky. For $\alpha(r) = \lambda$, $\lambda \in (0,1)$, these results were obtained in papers ([7,8]).

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Note: Figure translations are in progress. See original paper for figures.