ITERATIVE SOLUTION OF PROBLEMS OF LINEAR AND QUADRATIC PROGRAMMING

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Abstract

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MATHEMATICS

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ITERATIVE SOLUTION OF PROBLEMS OF LINEAR AND QUADRATIC PROGRAMMING

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In the present note, problems of linear and quadratic programming are solved by a method that is an extension of the method of steepest descent ([1]) to certain extremal problems among whose constraints there are inequalities.

Consider the linear programming problem:
find the minimum

$$\sum_{j=1}^{n} c_j x_j$$

under the conditions

$$x_j \geqslant 0, \quad j = 1, 2, \dots, n; \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \dots, m, \tag{2}$$

where $c = (c_1, c_2, \dots, c_n)$ and $x = (x_1, x_2, \dots, x_n)$ are vectors of the n-dimensional Euclidean space E_n; the rank of the matrix $\|a_{ij}\|$ is equal to m; $b = (b_1, b_2, \dots, b_m)$ is a vector from E_m. Let $T \subset E_n$ be the set of vectors satisfying conditions (1) and (2).

We shall call the **dual estimates** of $x \in T$ the system of numbers

$$u(x) = (u_1(x), u_2(x), \dots, u_m(x)),$$

satisfying the system of equations

$$\sum_{t=1}^{m} b_{st} u_t = d_s, \tag{3}$$

where

$$b_{st} = \sum_{j=1}^{n} x_j^2 a_{sj} a_{tj}, \quad d_s = \sum_{j=1}^{n} x_j^2 c_j a_{sj}, \quad s = 1, 2, \dots, m.$$

Remark. The vector $(u_1(x), u_2(x), \dots, u_m(x))$ is the solution of the problem: find

$$\min_{u \in E_m} \sum_{j=1}^{n} x_j^2 \left(\sum_{i=1}^{m} a_{ij} u_i - c_j \right)^2 .$$

Let $U(x)$ be the set of all solutions of system (3). Note that if $x_j > 0$ for $j = 1, 2, \dots, n$, then system (3) has a unique solution.

Set

$$\Phi(x) = \sum_{j=1}^{n} x_j^2 \left(\sum_{i=1}^{m} a_{ij} u_i(x) - c_j \right)^2 ,$$

$$s_j(x) = x_j^2 \left(\sum_{i=1}^{m} a_{ij} u_i(x) - c_j \right) .$$

We now give the algorithm for solving the linear programming problem. Let $x_j^0 > 0$ for $j = 1, 2, \dots, n$; compute

$$x_j^{k+1} = x_j^k + \lambda_k s_j(x^k)$$

for $j = 1, 2, \dots, n$,

$$\lambda_k = \frac{1}{\sqrt{\Phi(x^k)}}.$$

Theorem. If $\bar{x}$ is a limit point of the sequence $\{x^k\}$, then $\bar{x}$ is a solution of the linear programming problem.

The proof of the theorem follows from the validity of the following assertions, each of which is not difficult to establish.

1.
$$\sum_{j=1}^{n} a_{ij} s_j(x) = 0, \quad i = 1, 2, \dots, m.$$

2. $$\sum_{j=1}^{n} c_{js} j(x) = -\Phi(x).$$

3. If $x > 0$, then
$$x + s(x)/\sqrt{\Phi(x)} > 0.$$

4. The mapping $x \to U(x)$ is upper semicontinuous.

Suppose a quadratic programming problem is given: minimize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j - 2 \sum_{j=1}^{n} d_{jx} j$$

subject to

$$x_j \geq 0, \quad j = 1, 2, \dots, n;$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \dots, m,$$

where the matrix $\|b_{ij}\|$ is nonnegative definite, the rank of the matrix $\|a_{ij}\|$ is equal to m, $d = (d_1, d_2, \dots, d_n)$ and $x = (x_1, x_2, \dots, x_n)$ are vectors in E_n, and $b = (b_1, b_2, \dots, b_m)$ is a vector in E_m.

Set

$$r_j(x) = \sum_{i=1}^{n} b_{ij} x_i - d_j.$$

Construct a system of equations differing from (3) in that we replace c_j by $r_j(x)$. From the new system find

$$u(x) = (u_1(x), u_2(x), \dots, u_m(x)).$$

Let

$$\Phi(x) = \sum_{j=1}^{n} x_j^2 \left(\sum_{i=1}^{m} a_{ij} u_i(x) - r_j(x) \right)^2,$$

$$s_j(x) = x_j^2 \left(\sum_{i=1}^{m} a_{ij} u_i(x) - r_j(x) \right).$$

The following algorithm is proposed for solving the quadratic programming problem. Let $x^0 > 0$,

$$x^{k+1} = x^k + \lambda_k s(x^k);$$

if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} s_i(x^k) s_j(x^k) > 0,$$

compute

$$\lambda_k' = -\sum_{j=1}^{n} r_j(x^k) s_j(x^k) \Bigg/ \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} s_i(x^k) s_j(x^k).$$

If

$$\lambda_k' < 1/\sqrt{\Phi(x^k)},$$

set $\lambda_k = \lambda_k'$. In all other cases take

$$\lambda_k = 1/\sqrt{\Phi(x^k)}.$$

A theorem analogous to the one formulated above is valid.

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CITED LITERATURE

1. L. V. Kantorovich, DAN, **56**, 233 (1947).

Note: Figure translations are in progress. See original paper for figures.