LOCAL PROPERTIES OF NONRENORMALIZABLE THEORIES

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Abstract

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LOCAL PROPERTIES OF NONRENORMALIZABLE THEORIES

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1. One of the important features of quantum field theory, noted already by N. N. Bogolyubov (^{1}) and now attracting increasing attention, is that such properties of a theory as causality and locality are closely connected with the choice of the type of generalized functions of the theory. In this connection, the study of theories in which the physical quantities are not generalized functions over Schwartz space (generalized functions of moderate, i.e., polynomial, growth) becomes highly topical. Since the Schwartz space S is mapped into itself by the Fourier transform, it is clear that any departure beyond this space amounts to abandoning the requirement of moderate growth of generalized functions either in configuration space or in momentum space. The choice of the first of these paths does not lead us to the necessity of a radical reconstruction of the theory; on the contrary, in the usual formulation of axiomatic theory, instead of Schwartz space one often uses the space of basic functions with compact support belonging to the given type (the space D_x). However, by abandoning the requirement of polynomial growth in configuration space, we apparently do not introduce into consideration any new important class of physical theories. The degree of singularity at infinity in configuration space is not among the most essential characteristics of a theory. Therefore, bringing theories closer together according to the criterion of exponential growth in configuration space is not especially natural from the physical point of view.

In contrast, the asymptotic properties of physical quantities in momentum space are of primary interest for quantum field theory. It is known, in particular, that high (exponential) degrees of growth of generalized functions (matrix elements) in momentum space are the specific distinguishing feature of a quite definite class of physical theories—namely, nonrenormalizable theories. We conclude that, among all possible departures beyond Schwartz space, those which are connected with abandoning the requirement of polynomial growth of generalized functions in momentum space are of physical interest above all. In this case, for the definition of generalized functions with arbitrary degrees of growth, the space of basic functions with compact support (the space D_p) will evidently be required. In other words, only such a space makes it possible to consider the full set of nonrenormalizable theories, just as Schwartz space makes it possible to describe all renormalizable theories. In this—physical—sense the spaces S and D_p are complementary to one another and, taken together, sufficient for the description of all physical theories.

However, the basic functions from D_p in configuration space will be entire analytic functions. As such, they can no longer have compact support, and this makes the usual formulation of the locality postulate impossible. Therefore, in all works in which theories of this class were strictly investigated ($^{2-4}$), the problem was posed as follows: to determine what additional conditions must be imposed on the type and growth of generalized functions in order to single out the space

Φ (or a class of spaces) of basic functions satisfying the requirements: 1) Φ admits more than polynomial growth of generalized functions in momentum space (thereby excluding the spaces S and D_x); 2) Φ admits the usual formulation of the causality (locality) postulate. Such a problem, first posed by Meiman (^{2}), received its final solution in Jaffe' s work (^{4}).

We intend to show that, even without imposing any decisive restrictions on exponentially increasing generalized functions, and thus considering the full totality of nonrenormalizable theories, it is still possible to achieve the fulfillment of most physically essential properties and to construct the framework of a rigorous axiomatic description.

2. Thus, we consider the Wightman axiomatic theory in which the Heisenberg field operator in momentum space $\hat{A}(\tilde{f})$ is an operator-valued generalized function over the space D_p. Wightman's formalism is constructed over configuration space. The corresponding basic functions—the Fourier–Laplace transforms of functions from D_p—form the space $Z(C^4)$. This space (^{5}) contains all entire analytic functions $\psi(z)$ satisfying the inequalities

$$\left|z^k\psi(z)\right| \leq C_k(\psi)e^{a(\psi)|\operatorname{Im} z|}, \tag{1}$$

where

$$z=(z_1,\dots,z_n),\quad z^k=z_1^{k_1},\dots,z_n^{k_n},\quad |\operatorname{Im} z|=|\operatorname{Im} z_1|+\dots+|\operatorname{Im} z_n|. \tag{2}$$

Topologically, the space Z is an inductive limit, in the narrow sense, of the family of countably normed spaces $Z^{(a)}(C)$. Among the distinctive properties of this space, reflecting the specificity of the theory of holomorphic functions, are: 1) the absence in the space Z of functions with compact support ($D \cap Z = \emptyset$); 2) the impossibility of constructing a partition of unity from functions belonging to Z; 3) the expandability of all functions from Z in a Taylor series uniformly convergent in every finite domain.

Generalized functions in the space Z (field operators and their expectations $(\Psi, A(f_0), \dots, A(f_n)\Psi)$) will be called analytic functionals. The scalar field of such mathematical objects consists of complex, not real, numbers. Therefore, in order to obtain formulations of space-time properties of the theory (relativistic invariance, locality, etc.), we propose the following special procedure.

Since the space S' is dense in Z', there exists in Z' an everywhere dense set G' possessing the property that for every analytic functional $F \in G'$ there is a generalized function $F^* \in S'$ such that

$$\langle F, \psi \rangle = \left(F^*, \psi^{(R)}\right) \quad \text{for all } \psi(z) \in Z(C). \tag{3}$$

Here the functions $\psi^{(R)}(x)$ are the restrictions of analytic functions from Z to the real domain— "traces." It can be shown that the functional F^*, defined according to (3) on the set of functions $\psi^{(R)}(x)$ dense in S, can be continuously extended to all the remaining functions from S. Thus, the usual quantum-field requirements of a space-time character may be imposed on it in their standard formulations. In view of formula (3), all of them will simultaneously be requirements on the analytic functional $F \in G'$. Thus, using (3) and imposing space-time requirements on the functionals F^*, we directly obtain formulations of such requirements for analytic functionals from G'. However, these are implicit formulations, and later in each case one must solve the problem of passing from them to formulations directly in terms of the properties of the analytic functional F, without the mediation of F^*. Finally, having obtained such formulations for analytic functionals $F \in G'$, we shall adopt them as the expression of the space-time properties

for all analytic functionals in the space Z. Of course, the latter requirement is a certain "act of will," but it involves only a very small arbitrariness, if one takes into account that G' is dense in Z'.

3. Using the recipe constructed, we can formulate for an arbitrary theory over the space Z the system of Wightman postulates. In doing so, of course, not all theories will satisfy the full set of postulates; as usual, each physical requirement will also be a mathematical restriction excluding a certain type of theories. Listing the postulates below, we shall give their formulations (as a rule, in terms of vacuum expectations $\langle W, \psi \rangle$) only in those cases where they require a special derivation and differ in some way from the standard ones.

I. Relativistic invariance

$$\langle W, \psi \rangle = \langle W, \psi_{(a,\Lambda)} \rangle \tag{4}$$

or, equivalently,

$$W(\Lambda z_0 + a, \dots, \Lambda z_n + a) = W(z_0, \dots, z_n) \tag{4'}$$

for all real a and Lorentz transformations Λ from $L_+^\uparrow$.

Using the analyticity of the basic functions, it is not difficult to strengthen this condition substantially and to prove that the invariance condition (4) will also hold for all complex a and transformations Λ from the complex Lorentz group $\mathcal{L}_+$.

II. Existence and uniqueness of the vacuum.

III. Spectrality.

IV. Positive definiteness of the metric of the space of states.

V. Completeness (irreducibility of the algebra of field operators).

VI. Locality.

The formulation of locality is, naturally, the most difficult task. However, our scheme is applicable to this property as well. We shall start from the condition of locality on S, taken in the following form (which is easily obtained from the usual one): the Wightman functions on the space S, completely antisymmetrized in all arguments, have support, with respect to the variables $\xi_i = x_i - x_{i+1}$, in the light cone:

$$F(x_0, \dots, x_n) \equiv \sum_{P \in \Pi(n+1)} (-)^{N_P} W(P(x_0, \dots, x_n)); \quad \operatorname{supp} F \subset \overline{V}_\xi^n$$

(where $\Pi(n+1)$ is the permutation group of $(n+1)$ elements and N_P is the parity of the permutation P).

Hence, introducing the completely antisymmetrized Wightman functions $F(z_1, \dots, z_n)$ on Z, we have the following implicit formulation of locality:

$$\langle F(z), \psi(z) \rangle = (F^*(x), \psi^{(R)}(x)) \quad \text{for all} \quad \psi(z) \in Z(C^{4n}); \tag{5}$$

$$\operatorname{supp} F^* \subset \overline{V}^n \tag{6}$$

(it is assumed that z, x are translation-invariant variables).

The requirement expressed by these formulas must next be formulated directly in terms of the properties of the analytic functional F. One can prove that every analytic functional satisfying (5), (6) possesses a unique and continuous extension to the space $Z(\overline{V} + iR)$, consisting of functions $\psi(z)$, each of which is holomorphic in some neighborhood of the closed cylindrical domain $\overline{V} + iR$. In real timelike directions the functions from $Z(\overline{V} + iR)$ decrease faster than any power. In the complex domain they have first exponential order of growth:

$$|z^k \psi(z)| < C_\psi e^{a|\operatorname{Im} z|}$$

for all $z \in \overline{V} + iR$.

According to Martineau' s theorem (^{6}), in this case one says that the closed domain $\overline{V} + iR$ is a carrier of the analytic functional F from the space $Z'(C)$. Let us introduce for such a carrier the symbol A-supp. The resulting formulation of the locality property takes the following form: the completely antisymmetrized Wightman functions $F(z_1, \dots, z_n)$ possess

give, in the sense of the theory of analytic functionals, a carrier contained in the cylindrical domain $\bar{V} + iR$:

$$A\text{-}\operatorname{supp} F \subset \bar{V} + iR. \tag{7}$$

4. The final criterion of the reasonableness of the proposed formalism must, of course, be the system of its physical consequences. In general one may say that the system of postulates formulated apparently does not lead to any meaningless or contradictory consequences and, on the contrary, ensures the validity of most of the physically essential properties of ordinary theories. At the same time, since by construction condition (7) is weaker than the usual formulation of locality, not all results connected with locality remain valid. However, the following important properties still hold:

1) The strong asymptotic Haag—Ruelle condition is satisfied, and there exists a unitary (under the additional postulate of asymptotic completeness) scattering matrix S.
2) One of Borchers' main theorems remains valid: if two fields are mutually local (in the sense of condition (7)), then the corresponding scattering matrices are equal to one another.
3) For that part of nonrenormalizable theories in which the vacuum expectations are analytic functions in the tube and extended tube, one can formulate the condition of weak local commutativity and prove that it is a consequence of the locality condition in the form (7).

5. The just-mentioned class of theories over the space Z preserving the usual analyticity properties of the vacuum expectations possesses many properties which bring it substantially closer to ordinary theories. On this basis Schroer (^{7}) even divides all nonrenormalizable theories into two classes, assigning to the first of them precisely the theories with the usual analyticity properties. We shall now indicate a simple explicit characterization of such theories. According to a known theorem (^{8}), the Fourier—Laplace transform of a generalized function $\widetilde{W}(p) \in D'$ will be an analytic function in the tubular domain $\mathcal{T} = R - iV_+$ if and only if

$$(\widetilde{W}(p) e^{-py}) \in S' \quad \text{for all } y \in V_+. \tag{8}$$

It follows immediately from this that nonrenormalizable theories with the usual analyticity properties (Schroer' s theories of the first class) are singled out by condition (8)—a restriction on the degree of growth of the Wightman functions in momentum space. It can be proved that this restriction singles out the space of analytic functionals with real carrier (the space $Z'(R) \subset Z'(C)$). The elements of this space can be represented in the form of integrals along the real axis. Among them are generalized functions of the form

$$\sum_{n=0}^{\infty} c_n \delta^{(n)}(x - x_0)$$

under certain restrictions on the coefficients c_n, imposed by the requirement of convergence of the series in the topology of $Z'(R)$. It is easy to see (for example, taking in (8) $y = (\varepsilon, 000) \in V_+$) that condition (8) admits zero exponential order of growth in momentum space, and thus $Z'(R)$ is still substantially broader than the space $C'(R)$ considered by Jaffe (^{4}). Classes of functions close in their properties to $Z'(R)$ were studied in the one-dimensional case by Meiman (^{2}).

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