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ON THE DIRECT PROBLEM OF PLANE SYMMETRIC FLOW PAST A SMOOTH CONVEX PROFILE WITH A DETACHED SHOCK WAVE

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A. A. Nikol' skii and G. I. Taganov ([1]) showed that a potential flow in a local supersonic zone of a certain type (zone I) is destroyed if an arbitrary segment of the contour of the profile bounding this zone is straightened. In the present paper a related result is obtained for the case of symmetric flow past a smooth convex profile by a uniform supersonic stream with a detached shock wave.

Thus, we consider a plane adiabatic flow of an ideal gas behind a detached shock wave, possessing, in some neighborhood of the sonic point O on the contour of the profile, the properties of continuity of the velocity vector and of its first derivatives.

Let $\psi(x, y)$ be the stream function, vanishing on the critical streamline, which consists of a segment of the axis of symmetry and the contour of the profile. In view of the symmetry of the flow there exists a number $\psi_0 > 0$ such that, for $\psi(x, y) \leq \psi_0$, the entropy is a nonincreasing function of ψ.

Following ([1]), we shall call a supersonic region **zone I** if, from each point of the contour of the profile bounding it, both characteristics go out to the sonic line. In the flow past a strictly convex profile, zone I always exists, since the line of equal magnitude of velocity at the point of intersection with the contour makes an acute angle δ with the direction of the streamline, along which the velocity increases ([2]).

We shall call **zone III** a subregion of zone I in which $\psi(x, y) \leq \psi_0$, bounded by a segment of the contour, a characteristic of the first family, and a segment of the sonic line, at every point of which the angle δ is acute.

Theorem 1. *If one moves along the segment of the sonic line bounding zone III in such a way that the region of subsonic flow lies on the left, then the velocity vector turns monotonically clockwise (cf. the monotonicity law ([1])).*

Let $\partial/\partial s_1$ be the derivative in the direction $\mathbf{n}_1$ of the streamline, along which the velocity increases; $\partial/\partial s_2$, in the direction $\mathbf{n}_2$ normal to the streamline, along which ψ increases; $\partial/\partial\sigma$, in the direction $\vec{v}$ of the sonic line, when traversing it in such a way that the region of subsonic flow is situated on the left. We denote by: β the angle of inclination of the velocity vector to the axis of symmetry; $\lambda = v/a_*$ the velocity coefficient; M the Mach number; k the adiabatic exponent; R the gas constant; S the entropy; δ the angle between the vectors $\mathbf{n}_1$ and $\vec{v}$; the angles β and δ are measured counterclockwise.

From the equations of continuity and vorticity

$$\frac{\partial\beta}{\partial s_2} = (M^2-1)\frac{\partial\ln\lambda}{\partial s_1}, \qquad \frac{\partial\beta}{\partial s_1} = \frac{\partial\ln\lambda}{\partial s_2} + \frac{1}{kRM^2}\frac{dS}{ds_2} \tag{1}$$

we obtain, under the assumption that the derivative $\partial\lambda/\partial s_1$ is bounded, that

$$\frac{\partial\beta}{\partial\sigma} = \left(\frac{\partial\ln\lambda}{\partial s_2} + \frac{1}{Rk}\frac{dS}{ds_2}\right)\cos\delta = \left(-\frac{\partial\ln\lambda}{\partial s_1}\operatorname{ctg}\delta + \frac{1}{Rk}\frac{dS}{ds_2}\right)\cos\delta \quad \text{for } \lambda = 1.$$

On the segment of the sonic line under consideration, $dS/ds_2 \le 0$, $0 < \delta < \pi/2$, and therefore $\partial\beta/\partial\sigma \le 0$, which proves the theorem.

If the curvature of the profile contour at the sonic point O is not equal to zero, then in some neighborhood of the point O on the sonic line the strict inequality $\partial\beta/\partial\sigma < 0$ holds. If the first derivatives λ, β are continuous in some neighborhood of the point O, then the inequality $\partial\beta/\partial\sigma < 0$ also holds along any segment of the line $\lambda = \text{const}$ lying in a sufficiently small neighborhood of the point O. Hence there follows the one-to-one character of the mapping of this neighborhood onto the hodograph plane λ, β.

Indeed, if the mapping of any neighborhood of the point O onto the hodograph plane is not one-to-one, then a branching line passes through the point O. (A branching line is the edge of a fold that occurs under a non-one-to-one mapping of the flow into the hodograph plane.)

The branching line cannot coincide with the sonic line separating the region of subsonic flow from the region of supersonic flow, since otherwise, sufficiently close to the sonic line, there would exist at least two points situated on different sides of it with one and the same velocity vector. On the other hand, the branching line cannot intersect a segment of the line $\lambda = \text{const}$ situated sufficiently close to the point O, since otherwise at the intersection point the derivative $\partial\beta/\partial\sigma$ would change sign.

In the region of monotonicity of the entropy $S(\psi)$, the following definition is meaningful: the direction of traversal of a characteristic along which the entropy does not decrease is called **positive**.

Let, in the hodograph plane λ, β, the β-axis be directed vertically upward, and the λ-axis horizontally to the right.

For brevity we shall call an **epicycloid** the image in the λ, β plane of a characteristic of the potential flow.

From the relations along characteristics (by characteristics we mean only Mach lines)

$$\pm d\beta_{1,2} = \frac{\operatorname{ctg}\alpha}{\lambda}\, d\lambda + \frac{\sin 2\alpha}{2Rk}\, dS, \qquad \alpha = \arcsin\frac{1}{M} \tag{2}$$

it follows that

Lemma. *The image of a segment of a characteristic contained in the region of monotonicity of the entropy $S(\psi)$, in the plane λ, β, intersects the epicycloid of the same family no more than once; a segment of a characteristic of family I (II), drawn in the positive direction from the point of intersection with the epicycloid of the same family, is situated not below (not above) this epicycloid.*

Theorem 2. *A segment of the contour bounding, in zone III, an ε-neighborhood of a sonic point of the contour, whose mapping onto the hodograph plane is one-to-one, cannot contain a rectilinear portion; along this segment of the contour the estimate*

$$\left|\frac{\partial\lambda}{\partial s_1}\right| \leq \varkappa\lambda\tan\alpha,$$

holds, where $\varkappa = |\partial\beta/\partial s_1|$ is the curvature of the contour (cf. with (1)).

Since from each point of a segment of a streamline contained in zone III both characteristics pass to the sonic line in the negative direction, it follows from the lemma and from Theorem 1 that the image of any segment of a streamline contained in the supersonic subdomain of the ε-neighborhood does not intersect, in the hodograph plane, the triangle formed by the segments of the epicycloids drawn from each of its points to the sonic line. Therefore, for the angle of inclination φ of the image of the contour to the λ-axis, the estimate

$$|\tan\varphi| = |d\beta/d\lambda| \geq \operatorname{ctg}\alpha/\lambda,$$

holds, whence it follows that

$$|d\lambda/ds_1| \leq \varkappa\lambda\tan\alpha.$$

If there is a rectilinear segment on the contour, then its image in the λ, β plane is a segment of the straight line $\beta = \text{const}$ or a point. The former is impossible, since for $\lambda \neq 1$, $|d\beta/d\lambda| \leq \operatorname{ctg}\alpha/\lambda > 0$. Suppose

second. Then along the entire straight-line segment one has $\beta = \text{const}$ and $\lambda = \text{const}$. Solving the Cauchy problem with these initial data, we find that the flow inside the constructed characteristic triangle has rectilinear streamlines parallel to the straight-line segment of the contour, along which the velocity is constant, i.e. $\lambda = \lambda(\psi)$, where $\lambda(\psi)$ is a nonincreasing function. This means that the entire characteristic triangle is mapped in the hodograph plane onto the segment of the line $\beta =$ const, located outside the image of the ε-neighborhood of zone III, which is impossible.

Thus, straightening a certain portion of the contour situated in an ε-neighborhood of a sonic point of the contour, regardless of its length, leads either to such a deformation of the sonic line or of the characteristics that both characteristics issuing from any point of the rectilinear portion do not arrive at the sonic line, or to the formation of a jump, or to a violation of the one-to-one character of the mapping in the original ε-neighborhood, with the formation of a branching line intersecting the contour.

The first case apparently does not occur, since then the solution of the direct flow problem would not be unique, or there would be no continuous dependence of the solution on the boundary conditions.

We shall show that in the case of a loss of one-to-one correspondence in the ε-neighborhood, changes of the flow must arise which, in a certain sense, also do not depend on the length of the straightened portion. (We note that the straightening can be carried out so that the deformed contour will have arbitrarily many continuous derivatives.)

Consider the point W of intersection with the contour of the branching line that has formed. The Jacobian $\partial(\lambda, \beta)/\partial(x, y)$ becomes zero there if the flow remains smooth, or changes sign if a discontinuity of the first derivatives λ, β propagates from this point.

Consider the case when the Jacobian becomes zero at the point W. Taking into account that on the contour $dS/ds_2 = 0$, and using equations (1), we obtain at the point W

$$\frac{\partial(\lambda, \beta)}{\partial(x, y)} = \frac{\partial \lambda}{\partial \sigma_I} \frac{\partial \lambda}{\partial \sigma_{\mathrm{II}}} = 0,$$

where $d/d\sigma_I, d/d\sigma_{\mathrm{II}}$ are derivatives in the directions of the characteristics of families II and I.

Let us denote by a characteristic of family I $\xi =$ const (a characteristic of family II $\eta =$ const) if it passes through the point of the sonic line at which $\beta = \xi$ ($\beta = \eta$). In view of the fact that $\partial\beta/\partial\sigma \leq 0$ along the segment of the sonic line bounding zone III, such a definition is justified. Without loss of generality one may assume that at the point W, $\partial\lambda/\partial\sigma_I = 0$.

Since $d\sigma_I = d\xi/(\sin 2\alpha\sqrt{\xi_x^2 + \xi_y^2})$, we obtain, if at the point W $M \neq 1$ and $M \neq \infty$, that either $\partial\lambda/\partial\xi = 0$, or $\xi_x^2 + \xi_y^2 = 0$.

The equality $\xi_x^2 + \xi_y^2 = 0$ is impossible at an isolated point, since the characteristics do not have multiple singular points in the flow plane. If this equality held along the whole characteristic, then it would contradict the inequality $\partial\beta/\partial\sigma < 0$, which holds along the segment of the sonic line in the ε-neighborhood of the point O, since for $\lambda = 1$, $\partial\beta/\partial\sigma = \partial\xi/\partial\sigma = \partial\eta/\partial\sigma$. Thus we find that at the point W, $\partial\lambda/\partial\xi = 0$.

Integrating relations (2) along the characteristics of families I and II from the sonic line to an arbitrary point (ξ, η) of zone III, we obtain

$$\beta = \varphi(\lambda) + \int_{\eta_0}^{\eta} \frac{\sin 2\alpha}{2Rk}\frac{\partial S}{\partial \eta}\, d\eta + \xi = \varphi(\lambda) - \frac{1}{2Rk}\int_{\xi}^{\eta} q(\lambda)\sin\alpha \frac{dS}{d\psi}\frac{d\eta}{\sqrt{\eta_x^2+\eta_y^2}} + \xi,$$

$$-\beta = \varphi(\lambda) + \int_{\xi_0}^{\xi} \frac{\sin 2\alpha}{2Rk}\frac{\partial S}{\partial \xi}\, d\xi + \eta = \varphi(\lambda) - \frac{1}{2Rk}\int_{\eta}^{\xi} q(\lambda)\sin\alpha \frac{dS}{d\psi}\frac{d\xi}{\sqrt{\xi_x^2+\xi_y^2}} + \eta. \tag{3}$$

$$\varphi(\lambda) = \int_1^{\lambda} \frac{\operatorname{ctg}\alpha}{\lambda}\, d\lambda, \qquad q(\lambda) = \lambda\left(\frac{k+1}{2} - \frac{k-1}{2}\lambda^2\right)^{1/(k-1)}.$$

Differentiate relations (3) with respect to ξ and eliminate $\partial\beta/\partial\xi$. If the point (ξ, η) lies on the contour, we obtain

$$2\frac{\operatorname{ctg}\alpha}{\lambda}\frac{\partial\lambda}{\partial\xi} + F(\xi,\eta) = 0,$$

$$F(\xi,\eta) = 1 - \frac{1}{2Rk}\int_{\xi}^{\eta}\frac{\partial}{\partial\xi}\left(q(\lambda)\sin\alpha\frac{dS}{d\psi}\frac{1}{\sqrt{\eta_x^2+\eta_y^2}}\right)d\eta + \frac{dS}{d\psi}\frac{1}{\sqrt{\eta_x^2+\eta_y^2}}\Bigg|_{\eta=\xi(\lambda-1)}.$$

If, before the deformation is carried out, the derivative $\partial\lambda/\partial\xi$ is continuous, then the expression $F(\xi,\eta)$ is also continuous. At the sonic point of the contour $F(\xi,\eta) = 1$; therefore there exists a neighborhood of this point in which $F(\xi,\eta) > \rho$, $0 < \rho < 1$, where ρ is a certain constant. It follows that if the straightened segment is located sufficiently close to the sonic point of the contour, then the value of $F(\xi,\eta)$ will change at the point W by no less than ρ, independently of the length of this segment.

If, in the ε-neighborhood under consideration of the point O, the derivative $d^2S/d\psi^2$, as well as the first derivatives of λ, β, are bounded, then the integrand in $F(\xi,\eta)$ is bounded everywhere except at the point $\xi = \eta$, situated on the sonic line, since in this case the curvatures of the characteristics in the flow plane $\gamma_{\mathrm{I}}, \gamma_{\mathrm{II}}$ are bounded. (The derivative

$$\frac{\partial}{\partial \xi}\left(\sqrt{\eta_x^2+\eta_y^2}\right)$$

can be expressed in terms of γ_{I} and γ_{II}; $x_\xi^2+y_\xi^2 \neq 0,\ x_\eta^2+y_\eta^2 \neq 0$, since in the flow plane the characteristics of one family have no common points.) It follows from this that, when a certain segment of the profile contour is straightened, the value of $F(\xi,\eta)$ at the point W cannot change only through a rearrangement of the flow in a neighborhood of the point W commensurate with the length of the straightened segment. This means that if the problem of the external flow about a profile with a detached shock wave is well posed, then its solution, generally speaking, is not continuous.

The same conclusion is obtained also in the case when, at the point W, the Jacobian $\partial(\lambda,\beta)/\partial(x,y)$ changes sign without passing through zero.

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CITED LITERATURE

[1] A. A. Nikolsky, G. I. Taganov, PMM, **10**, issue 4 (1946).
[2] O. M. Belotserkovsky, PMM, **22**, issue 2 (1958).

Note: Figure translations are in progress. See original paper for figures.