MATHEMATICS

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Abstract

Full Text

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SPATIAL MAPPINGS WITH BOUNDED DISTORTION

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1. In what follows R^n denotes n-dimensional arithmetic Euclidean space. Let L be a linear mapping of the space R^n into itself such that $\det L \neq 0$. The mapping L transforms every sphere into an ellipsoid. The ratio of the largest semiaxis of this ellipsoid to its smallest semiaxis is called the **coefficient of distortion** of the mapping L and will be denoted below by $q(L)$.

A real-valued function $F(L)$, defined on the set of all square matrices of order n, is called a **conformal norm** if F is a norm in the vector space of matrices of order n and there exists a constant $\varkappa_F > 0$ such that for every matrix

$$F(L) \geq \varkappa_F |\det L|^{1/n},$$

with equality occurring if and only if $L = \alpha P$, where α is a number and P is an orthogonal matrix.

Set

$$q_F(L) = [F(L)]^n / \varkappa_F^n |\det L|.$$

The **conformal norm of a linear mapping** $L : R^n \to R^n$ is the conformal norm of its matrix.

For every conformal norm F the inequalities

$$q(L) \leq \varphi_1(F(L)/\varkappa_F |\det L|^{1/n}), \qquad F(L)/\varkappa_F |\det L|^{1/n} \leq \varphi_2(q(L)),$$

hold, where the functions φ_1 and φ_2 are such that as $x \to 1$, $\varphi_1(x) \to 1$, $\varphi_2(x) \to 1$.

2. Let U be a domain, i.e. a connected open set in R^n. We shall say that a mapping $f : U \to R^n$ belongs to the class W_n^1 if each of the coordinates $f_1, f_2, \dots, f_n$ of the vector function f has in U generalized first derivatives that are locally summable in U to the power n.

If $f : U \to R^n$ is a mapping of class W_n^1, then for almost all $x \in U$ the linear mapping

$$df_x(X) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) X_i, \quad X = (X_1, X_2, \dots, X_n),$$

is defined; we shall call it the **formal differential of the mapping** f at the point x. Set

$$\lambda(x,f)=\sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}(x)\right|^2,$$

$$J(x,f)=\det(df_x).$$

A mapping $f: U \to R^n$ is called a **mapping with bounded distortion** (abbreviated, a b.d. mapping) if $f \in W_n^1$ and there exists a constant K, $1 \leq K < \infty$, such that for almost all $x \in U$ the inequality

$$[\lambda(x,f)]^{n/2} \leq n^{n/2} K J(x,f). \tag{1}$$

holds.

If $f: U \to R^n$ is an o.i. mapping, then the function $q(df_x)$ is bounded in U. Its exact upper bound in U is called the **coefficient of distortion** of the mapping f and is denoted by $q(f,U)$. If F is a conformal norm, then the function $q_F(df_x)$ is also bounded in U. Its exact upper bound in U is called the **coefficient of distortion** of the mapping f in the norm F and is denoted by $q_F(f,U)$.

A special case of mappings with bounded distortion is formed by the so-called quasiconformal mappings.

A set $G \subset R^n$ is called a **compact domain** if G is compact, its open kernel is connected, and G is the closure of its open kernel. Let $f: U \to R^n$ be a continuous mapping, and let $G \subset U$ be a compact domain. Then $\mu(y, f|G)$, where $y \notin \operatorname{Fr} G$ ($\operatorname{Fr} G$ is the boundary of G), denotes the degree of the mapping $f: G \to R^n$ with respect to the point y.

Theorem 1. Every mapping $f: U \to R^n$ with bounded distortion is continuous and, for almost all $x \in U$,

$$f(x+X)=f(x)+df_x(X)+o(|X|).$$

Theorem 2. Let $f: U \to R^n$ be an o.i. mapping. Then, for every compact domain $G \subset R^n$ such that $\operatorname{mes}\operatorname{Fr} G = 0$, and for every bounded measurable function $u(y)$, the equality holds

$$\int_G u(f(x))\, J(x,f)\, dx = \int_{R^n} u(y)\, \mu(y,f|G)\, dy.$$

Let A be a compact set in R^n. One says that A is a **set of zero capacity** if, for every $\varepsilon > 0$ and every open set $B \supset A$, one can specify such an infinitely differentiable function φ that $\varphi(x)=0$ for $x \notin B$, $\varphi(x) \geq 1$ for $x \in A$, and

$$\int_{R^n}\sum_{i=1}^{n}\left|\frac{\partial \varphi}{\partial x_i}(x)\right|^n dx < \varepsilon.$$

Let A be an arbitrary set contained in the open domain U and closed relative to U. We shall say that A is a **set of zero capacity** if every compact subset of it has zero capacity.

Theorem 3. Let $f : U \to R^n$ be an o.i. mapping, not constant in U, such that, for every compact domain $G \subset U$, the function $\mu(y, f|G)$ is bounded. Then, for any set A of zero capacity, the set $f^{-1}(A)$ is also a set of zero capacity.

Theorem 4. Let $f : U \to R^n$ be an o.i. mapping, not constant in U, and such that, for every compact domain $G \subset U$, the function $\mu(y, f|G)$ is bounded. Then f is an open mapping.

For every point y and every compact domain $G \subset U$, $y \notin f(\operatorname{Fr} G)$, the set $f^{-1}(y) \cap G$ consists of no more than $\mu(y, f|G)$ elements.

Theorem 5. Let $\{f_m : U \to R^n\}$, $m = 1, 2, ...$, be a sequence of o.i. mappings such that the sequence $\{q(f_m, U)\}$ is bounded and, as $m \to \infty$, the mappings f_m converge to some mapping $f : U \to R^n$, the convergence being uniform on every compact set $A \subset U$. Then the limiting mapping f is a mapping with bounded distortion. Moreover, for every conformal norm F the inequality holds:

$$q_F(f, U) \leq \varliminf_{m\to\infty} q_F(f_m, U).$$

The following two theorems concern the so-called quasiconformal mappings. The class of quasiconformal mappings coincides with the class of topological mappings with bounded distortion.

Let $f : U \to R^n$ be a quasiconformal mapping. Then, for almost all $x \in U$, the linear mapping df_x is nondegenerate. Denote by

$E_f(x)$ is an ellipsoid which is transformed by the mapping df_x into the unit sphere of the space R^n.

Theorem 6. Let $f : U \to R^n$ and $g : U \to R^n$ be quasiconformal mappings of the domain U. Then, if for almost all $x \in U$ the ellipsoids $E_f(x)$ and $E_g(x)$ are similar, there exists a Möbius mapping $\varphi(y)$ such that

$$g(x) = \varphi[f(x)]$$

for all $x \in U$.

This theorem can be supplemented by a certain stability theorem.

Fix a bounded domain $U \subset R^n$ and a certain bounded quasiconformal mapping $f : U \to R^n$. Denote by $W_n^1(U, M)$ the set of all mappings g of class W_n^1 of the domain U into R^n such that for almost all $x \in U$, $|g(x)| \leq M$. For $g \in W_n^1$ put

$$\|g(x)\|_{W_n^1(G)} = \int_G \sum_{i=1}^n \left|\frac{\partial g}{\partial x_i}\right|^n dx,$$

where $G \subset U$,

$$V(g, U) = \int_U J(x, g)\, dx.$$

We also prescribe a certain conformal norm F. The ellipsoid $E_f(x)$ can be given by an equation of the form

$$|T(x, f)X| = \text{const},$$

where $T(x, f)$ is a positive definite symmetric matrix such that

$$\det T(x, f) = 1.$$

The coefficients of this matrix are measurable functions of the variable x. All eigenvalues of the matrix $T(x, f)$ lie in some interval (α, β), where $0 < \alpha < \beta$, and α and β depend only on the quantity $q(f, U)$. Put, for $g \in W_n^1$,

$$D(g; f, F) = \int_U \{F[dg_x \circ (T(x, f))^{-1}]\}^n dx.$$

Let us note that the mapping $f(x)$ gives the least value to the functional $D(g; f, F)$ in the class of mappings $g \in W_n^1$ coinciding with f in a neighborhood of the boundary of the domain U.

Theorem 7. For every $\varepsilon > 0$, every compact domain $G \subset U$, and every $M > 0$, there exists $\delta > 0$ such that, for every mapping $g \in W_n^1(U, M)$ such that

$$D[g; f, F] \leq \varkappa_F^n V(g, U)(1 + \delta),$$

one can indicate a Möbius mapping $\varphi(y)$ for which the inequality

$$\|g(x) - \varphi[f(x)]\|_{W_n^1(G)} < \varepsilon$$

holds.

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Note: Figure translations are in progress. See original paper for figures.