ELECTROMAGNETIC POTENTIALS AND THEIR GAUGE IN AN ANISOTROPIC, DISPERSIVE, AND MOVING MEDIUM

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Abstract

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ELECTROMAGNETIC POTENTIALS AND THEIR GAUGE IN AN ANISOTROPIC, DISPERSIVE, AND MOVING MEDIUM

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If the electromagnetic-field equation in four-dimensional form for the 4-tensors $F_{ik}(\mathbf{E}, \mathbf{B})$ and $\Phi_{ik}(\mathbf{D}, \mathbf{H})$ (see (1), § 33) is written for the Fourier amplitudes of these quantities, f_{ik} and φ_{ik}, respectively, then in a Galilean coordinate system with metric tensor

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

they have the form

$$\varphi^{ik} k_k = -i \frac{4\pi}{c} j^i, \tag{1}$$

$$\bar{f}^{ik} k_k = 0, \tag{2}$$

$$\varphi^{ik} = S^{ik}{}_{sr} \bar{f}^{sr}, \tag{3}$$

where $\bar{f}^{ik}$ is the pseudo-4-tensor dual to f_{ik} and related to it by the relation $\bar{f}^{ik} = {}^1/_2 \varepsilon^{ikmn} f_{mn}$; k_i is the wave 4-vector; j^i is the Fourier amplitude of the 4-vector of current density; $S^{ik}{}_{sr}$ is the material pseudo-4-tensor of permeability, antisymmetric in the upper pair and in the lower pair of indices, first introduced by I. E. Tamm ([2]). Here and below, indices denoted by letters of the Latin alphabet take the values $1, 2, 3, 0$ and number all coordinates of 4-space; summation is implied over twice repeated indices; ε^{iksr} are the contravariant components of the unit completely antisymmetric pseudo-4-tensor (([3]), §83), whose covariant components are such that $\varepsilon_{1230} = 1$ and $\varepsilon^{iksr} = -\varepsilon_{iksr}$.

Equation (2) can be satisfied by putting

$$\bar{f}^{ik} = iN^{iks}k_s, \tag{4}$$

where N^{iks} is a pseudo-4-tensor antisymmetric in all three indices. In the definition of N^{iks} there is an ambiguity: to N^{iks} one may, without changing $\bar{f}^{ik}$, add $T^{iksr}k_r$, where the pseudo-4-tensor T^{iksr} is arbitrary and completely antisymmetric, and therefore one may always put $T^{iksr} = T \cdot \varepsilon^{iksr}$, where T is a true 4-scalar. We introduce the 4-vector a_r, dual to the pseudo-4-tensor N^{iks} and being the Fourier amplitude of the 4-vector potential:

$$N^{iks} = \varepsilon^{iksr}a_r. \tag{5}$$

Since the pseudo-4-tensor N^{iks} is defined ambiguously, one may impose on the components N^{iks}, and consequently also on the components a_r, one entirely arbitrary scalar condition, which merely restricts the arbitrariness in the choice of the 4-scalar T.

Substituting (5) into (4), forming φ^{ik} with the aid of (3), and using (1), we obtain the equation for the 4-vector potential

$$T^{in}a_n = -\frac{4\pi}{c}j^i, \tag{6}$$

where

$$T^{in} = S^{ik}_{\cdot\cdot sr}\varepsilon^{srmn}k_k k_m = R^{ikmn}k_k k_m \tag{7}$$

is a true 4-tensor. Here the 4-tensor

$$R^{ikmn} = S^{ik}_{\cdot\cdot sr}\varepsilon^{srmn}, \tag{8}$$

has been introduced; it is antisymmetric in the first pair and in the second pair of indices and establishes the relation between φ^{ik} and f_{mn}

$$\varphi^{ik} = 1/2R^{ikmn}f_{mn}. \tag{9}$$

The relation inverse to (8) has the form

$$S^{ik}_{\cdot\cdot sr} = -1/4R^{ikmn}\varepsilon_{mnsr}. \tag{10}$$

For stationary media the components of the 4-tensor R^{ikmn} are expressed in terms of the components of the 3-tensors of the electric $\varepsilon^{\alpha\beta}$ and magnetic $\mu^{\alpha\beta}$ permeabilities

$$R^{\alpha\beta\gamma 0} = 0, \qquad R^{\alpha\beta\gamma\sigma} = \varepsilon^{\alpha\beta\lambda}\mu^{-1}_{\lambda\nu}\varepsilon^{\nu\gamma\sigma}, \qquad R^{\alpha 0\beta\gamma} = 0, \qquad R^{\alpha 0\sigma 0} = -\varepsilon^{\alpha\sigma}. \tag{11}$$

Indices denoted by letters of the Greek alphabet take the values 1, 2, 3 and number the spatial coordinates. In three-dimensional space the unit completely antisymmetric pseudo-3-tensor $\varepsilon^{\alpha\beta\gamma}$ and $\varepsilon_{\alpha\beta\gamma}$ is related to the analogous 4-tensor in four-dimensional space by the relations

$$\varepsilon^{\alpha\beta\gamma} \equiv \varepsilon^{\alpha\beta\gamma 0}, \qquad \varepsilon_{\alpha\beta\gamma} \equiv \varepsilon_{\alpha\beta\gamma 0}.$$

Introduce the 4-tensor

$$G_{ik} = 1/4\varepsilon_{ibsr}k_l R^{lspr}k_a R^{abmt}\varepsilon_{mtpk} \tag{12}$$

and multiply both sides of equation (6) by it, summing over the index i. Then we obtain

$$T^{in}a_n G_{ik} = -\frac{4\pi}{c}j^i G_{ik}$$

or

$$C_k{}^n a_n = -\frac{4\pi}{c}j^i G_{ik}, \tag{13}$$

where

$$C_k{}^n = G_{ik}T^{in}. \tag{14}$$

Using (7) and (8), for $C_k{}^n$ we obtain

$$C_k{}^n = A^{pmt}_{vwg}\varepsilon_{mtpk}\varepsilon^{vwgn}, \tag{15}$$

where the 4-tensor

$$A^{pmt}_{vwg} = 1/4\varepsilon_{ibsr}k_l R^{lspr}k_a R^{abmt}S^{if}_{\cdot\cdot vwk}k_f k_g, \tag{16}$$

has been introduced; it is antisymmetric in the first pair of lower and in the second pair of upper indices. By virtue of the identity

$$\varepsilon_{mtpk}\varepsilon^{vwgn} \equiv \varepsilon_{mtps}\varepsilon^{vwgs}\delta^n_k + \varepsilon_{mtps}\varepsilon^{vwsn}\delta^g_k + \varepsilon_{mtps}\varepsilon^{vsgn}\delta^w_k + \varepsilon_{mtps}\varepsilon^{swgn}\delta^v_k \tag{17}$$

equality (15) takes the form

$$C_k{}^n = A^{pmt}_{vwg}\varepsilon_{mtps}\varepsilon^{vwgs}\delta^n_k + A^{pmt}_{vwk}\varepsilon_{mtps}\varepsilon^{vwsn} - 2A^{pmt}_{kwg}\varepsilon_{mtps}\varepsilon^{nwgs}. \tag{18}$$

The last term in (18) can be transformed using the relation

$$\varepsilon_{mtps}\varepsilon^{nwgs} = -\left(\delta^n_m\delta^w_t\delta^g_p + \delta^w_m\delta^g_t\delta^n_p + \delta^g_m\delta^n_t\delta^w_p - \delta^n_m\delta^g_t\delta^w_p - \delta^w_m\delta^n_t\delta^g_p - \delta^g_m\delta^w_t\delta^n_p\right) \tag{19}$$

and the symmetry properties of the 4-tensor A^{pmt}_{vwg} noted above. As a result we obtain

$$-2A^{pmt}_{kwg}\varepsilon_{mtps}\varepsilon^{nwgs} = -4A^{gwn}_{kwg} + 4A^{nwg}_{kwg} + 4A^{wng}_{wkg}. \tag{20}$$

It is not difficult to verify that the first term in (20) is equal to zero

$$-4A^{gwn}_{kwg} = \varepsilon_{ibsr}k_l R^{lsrg}k_g k_a R^{abwn}k_f S^{if\cdot\cdot}_{\cdot\cdot kw} = 0. \tag{21}$$

For this, instead of $k_l R^{lsrg}k_g$, one must substitute, in accordance with (8), $k_l S^{ls\cdot\cdot}_{\cdot\cdot vm}\varepsilon^{vmrg}k_g$, and expand the product $\varepsilon_{ibsr}\varepsilon^{vmrg}$ by formula (19); then in each of the 6 terms there will be present the product of a symmetric 4-tensor of rank 2 with a 4-tensor antisymmetric in the same indices.

It can be shown that the 3rd term in (20)

$$4A^{wng}_{wkg} = \varepsilon_{ibsr}k_l R^{lswr}k_f S^{if\cdot\cdot}_{\cdot\cdot wk}k_a R^{abng}k_g = -C^n_k. \tag{22}$$

To convince oneself of this, note that the last three factors in (22) form, according to (7), the 4-tensor T^{bn}, and, if the factor $S^{if\cdot\cdot}_{\cdot\cdot wk}$ is transformed by formula (10), then all the factors standing before T^{bn} form, according to (12), G_{bk}, and on the basis of (14)

$$-G_{bk}T^{bn} = -C^n_k.$$

Let us transform the second term in (20)

$$4A^{nwg}_{kwg} = \varepsilon_{ibsr}k_l R^{lsnr}k_a R^{abwg}k_g k_f S^{if\cdot\cdot}_{\cdot\cdot kw}. \tag{23}$$

If in (23) we use the equality $R^{abwg} = S^{ab\cdot\cdot}_{\cdot\cdot vm}\varepsilon^{vmwg}$ (see (8)), and then the identity analogous to (17):

$$\varepsilon_{ibsr}\varepsilon^{vmwg} \equiv \varepsilon_{pbsr}\varepsilon^{vmwp}\delta^g_i + \varepsilon_{ibsp}\varepsilon^{vmwp}\delta^g_r, + \varepsilon_{ibpr}\varepsilon^{vmwp}\delta^g_s + \varepsilon_{ibsp}\varepsilon^{vmwp}\delta^g_r, \tag{24}$$

then it is obvious that, of the four terms formed in (23), three turn into zero because in each there is present the product of a 4-tensor symmetric in two indices with a 4-tensor antisymmetric in the same indices: in one, $k_a k_b S^{ab\cdot\cdot}_{\cdot\cdot vm}$, in another, $k_l k_s R^{lsnr}$, and in the third, $k_i k_f S^{if\cdot\cdot}_{\cdot\cdot kw}$. Then

$$4A^{nwg}_{kwg} = \varepsilon_{ibsp} k_a S^{ab\cdot\cdot}_{\cdot\cdot vm} \varepsilon^{vmwp} k_f S^{if\cdot\cdot}_{\cdot\cdot lw} k_l R^{lsnr} k_r.$$

If one notes that, according to (8), $S^{ab\cdot\cdot}_{\cdot\cdot vm}\varepsilon^{vmwp} = R^{abwp}$, then the resulting expression does not differ from that which stands in (22) after the first equals sign. Consequently,

$$4A^{nwg}_{kwg} = -C^n_k. \tag{25}$$

Taking into account (21) and (25), formula (18) takes the form

$$C^n_k = V\delta^n_k + k_k P^n, \tag{26}$$

where the 4-scalar $V = \frac{1}{3} A^{pmt}_{wwg}\varepsilon_{mtps}\varepsilon^{vwgs}$ and the 4-vector P^n are introduced, the latter having the form

$$P^n = \frac{1}{3} G_{is} R^{ifsn} k_f. \tag{27}$$

Substituting (26) into (13), we obtain

$$V\delta^n_k a_n + k_k P^n a_n = -\frac{4\pi}{c} j^i G_{ik}. \tag{28}$$

Since one scalar condition may be imposed on a_n, choosing it in the form $P^n a_n = 0$, we obtain from (28) the solution for the 4-vector potential

$$a_k = -\frac{4\pi}{c}\frac{j^i G_{ik}}{V}. \tag{29}$$

The equality $P^n a_n = 0$ is a generalization of the Lorentz condition in a material medium that is anisotropic and dispersive with respect to its electric and magnetic properties. Using (11), we obtain the components of the 4-vector P^n for stationary media:

$$P^0 = \frac{1}{3} G_{is} R^{ifs0} k_f = \frac{1}{2} k_0 T^{00} \varepsilon^{\sigma\nu} \mu^{-1}_{\sigma\nu} - \frac{1}{2} k_0 g^\alpha \mu^{-1}_{\alpha\nu} p^\nu - k_0^3 |\varepsilon^{\alpha\beta}|;$$

$$P^\alpha = \frac{1}{3} G_{is} R^{ifs\alpha} k_f =$$

$$= -\frac{1}{2} \frac{k_\sigma \mu^{\sigma\nu} k_\nu}{|\mu^{\alpha\beta}|} p^\alpha + \frac{1}{2} k_0^2 \varepsilon^{\sigma\nu} \mu^{-1}_{\sigma\nu} p^\alpha - \frac{1}{2} k_0^2 \mu^{-1}_{\sigma\nu} p^\nu \varepsilon^{\sigma\alpha} - \frac{1}{2} T^{00} \frac{k_\sigma \mu^{\sigma\alpha}}{|\mu^{\alpha\beta}|}, \qquad (30)$$

where the notations $p^\alpha = k_\beta \varepsilon^{\beta\alpha}$, $g^\alpha = \varepsilon^{\alpha\beta} k_\beta$, $T^{00} = k_\sigma \varepsilon^{\sigma\nu} k_\nu$ have been introduced; $|\mu^{\alpha\beta}|$ and $|\varepsilon^{\alpha\beta}|$ are determinants composed of the components of the 3-tensors $\mu^{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$, respectively.

For an isotropic medium the equality $P^n a_n = 0$ becomes

$$\frac{\varepsilon}{\mu} \left(\varepsilon k_0^2 - \frac{k_\alpha k^\alpha}{\mu} \right) (k^\alpha a_\alpha - \varepsilon\mu k_0 a_0) = 0,$$

which is satisfied under the condition

$$k^\alpha a_\alpha - \varepsilon\mu k_0 a_0 = 0,$$

or, taking into account that $k_0 = -\omega/c$, $a_0 = -\varphi$,

$$\mathbf{k} \cdot \mathbf{a} - \frac{\omega\varepsilon\mu}{c} \varphi = 0. \qquad (31)$$

Since $\mathbf{a}$ is the Fourier amplitude of the 3-vector potential of the electromagnetic field $\mathbf{A}$, and φ is the Fourier amplitude of the scalar potential Φ, equality (31) obviously corresponds to the condition

$$\operatorname{div} \mathbf{A} + \frac{\varepsilon\mu}{c} \frac{\partial \Phi}{\partial t} = 0.$$

The solution (29) was first obtained by one of the authors by a more cumbersome route without formulating the gauge condition for the 4-potential [4]. Of course, the solution presented here, based on the use of the 4-tensor G_{ik}, and the gauge condition could have been found only after carrying out the aforementioned cumbersome solution, since the method of constructing the 4-tensor G_{ik} is not evident in advance.

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Note: Figure translations are in progress. See original paper for figures.