PROBLEMS OF BEST APPROXIMATION BY ELEMENTS OF A CONVEX SET AND SOME PROPERTIES OF SUPPORT FUNCTIONALS

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PROBLEMS OF BEST APPROXIMATION BY ELEMENTS OF A CONVEX SET AND SOME PROPERTIES OF SUPPORT FUNCTIONALS

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1. In this note we consider a number of problems of best approximation by elements of a convex subset of a B-space. The criteria for best approximation formulated here are closely connected with certain assertions on support functionals, which are of independent interest. These assertions are also included in the note.

2. Let G be a convex subset of a real B-space E; let $x_1, x_2, \dots, x_n$ be certain elements of E; let $\varphi(t_1, t_2, \dots, t_n) = \varphi(t)$ be a convex function defined on the n-dimensional space Q_n and possessing the monotonicity property: $\varphi(t') \geq \varphi(t'')$, if $t' \geq t''$. A sequence $\{x^{(k)}\}$ of elements $x^{(k)} \in G$ will be called a sequence of best, in the sense of φ, approximations of the aggregate of points x_i, $i = 1, 2, \dots, n$, by elements of G, if

$$\lim_{k\to\infty} \varphi(|x_1 - x^{(k)}|,\ |x_2 - x^{(k)}|, \dots,\ |x_n - x^{(k)}|) =$$

$$= \inf_{x\in G} \varphi(|x_1 - x|,\ |x_2 - x|, \dots,\ |x_n - x|).$$

We shall denote the problem of finding such a sequence by $A(x_1, x_2, \dots, x_n; G; \varphi)$. If the sequence of best approximation $\{x^{(k)}\} = \{x^*\}$ is stationary, then x^* will be called an element of best, in the sense of φ, approximation of the points x_i, $i = 1, 2, \dots, n$, by elements of G.

For what follows we shall need a certain generalization of the concept of a support functional. Let S be a subset of the B-space E; let $f(x)$ be a convex functional, defined and continuous on S. A linear functional $l \in E^*$ will be called supporting for f at the generalized point $\{x^{(k)}\}$, $x^{(k)} \in S$, relative to S, if

$$\tilde{f}_s(l) = \inf_{x\in S} [f(x) - l(x)] = \lim_{k\to\infty} [f(x^{(k)}) - l(x^{(k)})] > -\infty. \tag{1}$$

In the case when the sequence $\{x^{(k)}\}$ is stationary, the concept introduced becomes the known concept of a support functional at a point. We shall denote by $\Omega(f, S, \{x^{(k)}\})$ the set of functionals $l \in E^*$ supporting f at the generalized point $\{x^{(k)}\}$ relative to S. The assertion given below is a criterion for best approximation in the problem $A(x_1, x_2, \dots, x_n; G; \varphi)$.

Theorem 1. The sequence $\{x^{(k)}\}$, $x^{(k)} \in G$, is a sequence of best, in the sense of φ, approximations of the aggregate of points $x_1, x_2, \dots, x_n$ by elements of G if and only if there exist linear functionals $g_i \in E^*$, $|g_i| \leq 1$, $i \in I$, and a nonnegative vector $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega(\varphi, Q_n - \{t^{(k)}\})$, such that

$$\lim_{k\to\infty} \left[\, |x_i - x^{(k)}| - g_i(x^{(k)} - x_i) \,\right] = 0, \qquad i \in I; \tag{2}$$

$$\inf_{x\in G} \left[\sum_{i\in I} \alpha_i g_i(x) \right] = \lim_{k\to\infty} \left[\sum_{i\in I} \alpha_i g_i(x^{(k)}) \right], \tag{3}$$

where $t^{(k)} = (|x_1 - x^{(k)}|,\ |x_2 - x^{(k)}|, \dots,\ |x_n - x^{(k)}|)$, $I = \{i : \alpha_i > 0\}$.

Let us note two special cases of the problem $A(x_1, x_2, \dots, x_n; G; \varphi)$ for $\varphi(t) = \varphi_1(t) = \sum_{i=1}^n a_i t_i,\ a_i > 0$, and $\varphi(t) = \varphi_2(t) = \max_{1\leq i\leq n} t_i$. The criterion for best approximation in the problem $A(x_1, x_2, \dots, x_n; G; \varphi_1)$ consists, obviously, in satisfying the requirements (2), (3), where $g_i \in E^*,\ |g_i| \leq 1,\ i \in I = (1, 2, \dots, n)$. Under the assumption of stationarity of the sequence $\{x^{(k)}\}$, this criterion was obtained in (4). The criterion for best approximation in the problem $A(x_1, x_2, \dots, x_n; G; \varphi_2)$, as follows from Theorem 1, again consists in satisfying the requirements (2), (3), where

$$g_i \in E^*,\ |g_i| \leq 1,\ i \in I \subset M = \{i : \lim_{k\to\infty} |x_i - x^{(k)}| =$$

$$= \lim_{k\to\infty} \max_{1\leq i\leq n} |x_i - x^{(k)}|\}; \quad \alpha_i > 0,\ i \in I,\ \sum_{i\in I} \alpha_i = 1.$$

Remark 1. In works devoted to problems of best approximation by elements of a convex set (we note (1–3), in which the case $n = 1$ was studied, and (4), where the problem $A(x_1, x_2, \dots, x_n; G; \varphi_1)$ was considered), criteria for best approximation were established under the assumption that an element of best approximation exists. However, whereas a sequence of best approximation always exists, the existence of an element of best approximation can be guaranteed only in comparatively rare cases. Therefore Theorem 1 and the corollaries following from it have a much broader range of applications than the corresponding criteria for an element of best approximation.

3. We now consider problems of best approximation in which the set being approximated consists of an infinite number of elements. Let K be a bounded

Borel subset of the space E, and let $\alpha(F)$ be a nonnegative countably additive function defined on the σ-algebra $\{F\}$ of Borel subsets of K, $\alpha(K) < \infty$. Put

$$f(x) = \int_{y \in K} |y - x| \, d\alpha, \quad x \in E. \tag{4}$$

By $A_\alpha(K; G)$ we shall denote the problem of best approximation, in the sense of α, of the set K by means of elements of the convex set G, $G \subset E$, which consists in finding a sequence $\{x^{(k)}\}$, $x^{(k)} \in G$, minimizing the functional (4) on G. All assertions of this section are established under the assumption that there exists an element of best approximation x^* of the corresponding problem, for which $f(x^*) = \inf_{x \in G} f(x)$, $x^* \in G$.

Theorem 2. *Let the space E be separable. A point $x^* \in G$ is an element of best approximation in the problem $A_\alpha(K, G)$ if and only if there exists a family of linear functionals $g_y \in E^*$, $y \in K$, having the following properties:*

$$|g_y| \le 1, \quad g_y(x^* - y) = |y - x^*|, \quad y \in K; \tag{5}$$

$$g_y(x) \text{ is an } \alpha\text{-measurable function of } y \in K$$

$$\text{for every fixed } x \in E; \tag{6}$$

$$\inf_{x \in G} \int_{y \in K} g_y(x) \, d\alpha = \int_{y \in K} g_y(x^*) \, d\alpha. \tag{7}$$

The problem $A_\alpha(K; G)$ is a continual analogue of the problem $A(x_1, x_2, \dots, x_n; G; \varphi_1)$. Below we study a continual analogue of the more general problem $A(x_1, x_2, \dots, x_n; G; \varphi)$.

Let $\varphi(\gamma)$ be a convex real functional, defined and continuous on the normed space $C(K)$, $K \subset E$, of real functions $\gamma(y)$, $y \in K$, continuous and bounded on K, where $|\gamma| =$

$$= \sup_{x \in K} |\gamma(x)|.$$

We shall assume that the functional $\varphi(\gamma)$ has the monotonicity property:

$$\varphi(\gamma') \ge \varphi(\gamma''), \qquad \text{if } \gamma'(y) \ge \gamma''(y) \quad \text{for } y \in K.$$

Put

$$f(x) = \varphi(\gamma_x), \qquad \text{where } \gamma_x(y) = |y - x|. \tag{8}$$

By $A(K;G;\varphi)$ we denote the problem of best approximation, in the sense of φ, of the set K by elements of the convex set G, which consists in finding a sequence $\{x^{(k)}\}$, $x^{(k)} \in G$, minimizing the functional (8) on G.

Theorem 3. Let the space E be separable, and let $K \subset E$ be compact. A point $x^* \in G$ is an element of best approximation in the problem $A(K;G;\varphi)$ if and only if there exist a family of functionals $g_y \in E^*$, $y \in K$, and a function $\alpha(F)$, $\alpha \in V(K) = C^*(K)$, $\alpha \geq 0$, such that the conditions (5)–(7) of Theorem 2 are satisfied and the linear functional generated by $\alpha(F)$ is contained in $\Omega(\varphi, C(K), \gamma_{x^*})$.

Obviously, under the assumption that K is compact, Theorem 2 is a special case of Theorem 3.

Let us note a special case of the problem $A(K;G;\varphi)$ when

$$\varphi(\gamma) = \varphi_2(\gamma) = \max_{y \in K} \gamma(y).$$

From Theorem 3 it follows:

Theorem 4. Let E be separable, and let K be compact. A point $x^* \in G$ is an element of best approximation in the problem $A(K;G;\varphi_2)$ if and only if there exist a family of functionals $g_y \in E^*$, $y \in K_0$, and a function $\alpha \in V(K_0)$, $\alpha \geq 0$, $\alpha(K_0) = 1$, satisfying conditions (5)–(7), where in place of K one has

$$K_0 = \left\{ y : \ |y - x^*| = \max_{y' \in K} |y' - x^*|, \ y \in K \right\}.$$

Remark 2. If K consists of a countable number of elements, then the assumption that E is separable in Theorems 2–4 may be dropped.

Remark 3. Theorems 1–4 are easily carried over to the case of a complex space E. The formulations of the complex analogues of Theorems 1–4 differ only in that, in conditions (3) and (7), $g_i(x)$ and $g_y(x)$ are replaced respectively by $\mathrm{Re}\{g_i(x)\}$ and $\mathrm{Re}\{g_y(x)\}$.

4. The proofs of Theorems 1–4 are based on a number of propositions concerning support functionals. Since these propositions may be used in various extremal problems, we give their formulations. All functionals considered in this section are assumed to be defined and continuous on an open convex subset S of a real B-space E.

Theorem 5. If $f(x)$ is a functional convex on S,

$$v = \inf_{x \in R} f(x) = \lim_{k \to \infty} f(x^{(k)}), \qquad x^{(k)} \in R \subset S,$$

then there exists a functional $g \in \Omega(f, S, \{x^{(k)}\})$ such that

$$v = \inf_{x \in R} g(x) + \tilde{f}_s(g) = \lim_{k \to \infty} g(x^{(k)}) + \tilde{f}_s(g).$$

Theorem 6. If

$$f(x)=\sum_{i=1}^{n}\alpha_i f_i(x), \qquad \alpha_i>0,$$

and $f_i(x)$ is a functional convex on S ($i=1,2,\dots,n$), $x^{(k)}\in S$, then $\Omega(f,S,\{x^{(k)}\})\neq\emptyset$ if and only if

$$\Omega(f_i,S,\{x^{(k)}\})\neq\emptyset, \qquad i=1,2,\dots,n, \tag{9}$$

and, when (9) is satisfied,

$$\Omega(f,S,\{x^{(k)}\})=\sum_{i=1}^{n}\alpha_i\Omega(f_i,S,\{x^{(k)}\}). \tag{10}$$

Formula (10), under the assumption of stationarity of the sequence $\{x^{(k)}\}=\{x^*\}$ and under a number of additional requirements on the functionals f_i, is contained in (5).

Let K be a Hausdorff topological space; $f(x,t)=f_t(x)$, $t\in K$, a family of functionals convex on S and possessing the following properties: a) $f(x,t)$ depends continuously on $t\in K$ for every fixed $x\in S$; b) for some $\gamma>0$ the functional $f(x,t)$ is uniformly continuous on the set

$$Q_{x^*}=\{(x,t):\ |x-x^*|\le\gamma,\ x\in S,\ t\in K\}, \quad x^*\in S;$$

c) $\sup_{t\in K}|f(x,t)|<\infty$ for all $x\in S$. Denote by $\alpha(F)$ a nonnegative countably additive function defined on the σ-algebra $\{F\}$ of Borel subsets of K, $\alpha(K)<\infty$. Put

$$f(x)=\int_{t\in K}f(x,t)\,d\alpha.$$

Obviously, $f(x)$ is defined on the set S.

Theorem 7. *Let E be separable. The set $\Omega(f,S,x^*)$, $x^*\in S$, consists of those and only those functionals $g\in E^*$ for which there is a representation*

$$g(x)=\int_{t\in K}g_t(x)\,d\alpha,$$

where $g_t\in\Omega(f_t,S,x^*)$, $t\in K$; $g_t(x)$ is an α-measurable function of $t\in K$ for every fixed $x\in E$.

Remark 4. The assumption that E is separable may be removed if K consists of a countable number of elements.

Let $\Phi(x)$ be an operator acting from $S\subset E$ into the real B-space E_1, and let $\varphi(y)$ be a real functional defined on E_1. Suppose that $\Phi(x)$ is a convex operator relative to some convex cone $G_1\subset E_1$, i.e.

$$\delta\Phi(x')+(1-\delta)\Phi(x'')-\Phi(\delta x'+(1-\delta)x'')\in G_1,$$

$$\text{if } x', x'' \in S,\ 0 \le \delta \le 1;$$

$\varphi(y)$ is a continuous convex functional possessing the property of monotonicity relative to the cone G_1, i.e.

$$\varphi(y') \ge \varphi(y''), \quad \text{if } y' - y'' \in G_1.$$

Theorem 8. *If $f(x) = \varphi(\Phi(x))$, $x_k \in S$, $k = 1, 2, \ldots$, then*

$$\Omega(f, S, \{x^{(k)}\}) = \bigcup_{g \in \Omega'} \Omega(f_g, S, \{x^{(k)}\}),$$

where $f_g(x) = g(\Phi(x))$, $g \in E_1^*$; $\Omega' = \Omega(\varphi, E_1, \{\Phi(x^{(k)})\})$.

The proof of Theorem 1 is based on Theorems 5, 6, and 8. Theorem 2 follows from Theorems 5 and 7. The proof of Theorem 3 consists in the successive application of Theorems 5, 8, and 7.

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Note: Figure translations are in progress. See original paper for figures.