APPLICATION OF CHEBYSHEV POLYNOMIALS ORTHONORMALIZED ON A SYSTEM OF EQUALLY SPACED POINTS TO THE SOLUTION OF INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract

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APPLICATION OF CHEBYSHEV POLYNOMIALS ORTHONORMALIZED ON A SYSTEM OF EQUALLY SPACED POINTS TO THE SOLUTION OF INTEGRAL EQUATIONS OF THE FIRST KIND

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Consider the Fredholm integral equation of the first kind

$$\int_a^b K(x,y)z(y)\,dy = u(x), \qquad a \leqslant x \leqslant b. \tag{1}$$

The unknown function $z(y)$ could be computed by solving the system

$$\sum_{s=0}^{n} K_{r,s} z_s = u_r,$$

where $r, s = 0(1)n, \quad u_r = u(x_r), \quad z_s = z(y_s), \quad K_{rs} = K(x_r, y_s)\Delta y,$

$$\tag{2}$$

but this system is ill-conditioned, which leads to a large scatter of the values z_s, not allowing one to construct the desired smooth function $z(y)$.

The difficulties that arise here are connected with the ill-posedness of the problem posed. These difficulties can be overcome to a considerable degree if one takes into account the profound analogy between the measurement of any physical quantity and the solution of the integral equation (1): $z(y)$ plays the role of the measured function, $u(x)$ of the readings of the measuring instrument, and the scatter of the values z_s is analogous to measurement errors.

As is known, Legendre proposed a method of smoothing measurement errors based on approximating the measured function by polynomials. Since the values of the function are subject to random errors, it is inexpedient to introduce these errors into the approximating polynomial; therefore the polynomial should not be made to pass through these points, and it is much more effective to take a polynomial of not very high degree that would approximate the measurement results "on the average," minimizing the sum of squares of the deviations. Gauss gave a theoretical-probabilistic justification of this method, which received the name of the method of least squares. This method, using the entire set of function values, makes it possible, without copying random scatter, to find a polynomial of a prescribed degree that agrees best of all with the measured values (namely, the sum of squared deviations is minimal). The coefficients of the polynomials are found from the system of linear Gauss equations, which, as a rule, is ill-conditioned.

For the case when the desired function is measured at equally spaced points, Chebyshev proposed a method for determining the coefficients of the approximating polynomial that does not require solving the Gauss system and is based on the use of the polynomials he introduced, orthonormalized on a system of equally spaced points. We shall use these Chebyshev polynomials to solve system (2). Tables of values of these polynomials are given in the work (^{1}).

The Chebyshev polynomials $P_{m,n}(x)$ satisfy the relation

$$\sum_{x=0}^{n} P_{j,n}(x) P_{k,n}(x) = \delta_{jk}. \tag{3}$$

Fix n and approximate the unknown function $z(y)$ by a polynomial of degree m

$$Z_m(y_s) = \sum_{k=0}^{m} b_k P_{k,n}(s), \tag{4}$$

where

$$b_k = \sum_{s=0}^{n} z_s P_{k,n}(s). \tag{5}$$

Direct use of these relations is expedient when the scatter of the values z_s is small. If, however, the scatter is large, it makes no sense to find z_s from system (2) and then approximate the function $z(y)$ by the polynomial (4); rather, one should determine the coefficients b_k at once (computationally this is always more advantageous). In order to obtain a system with the number of equations equal to the number of unknowns, we shall approximate the function $u(x)$ by a polynomial of the same degree

$$U_m(x_r) = \sum_{j=0}^{m} c_j P_{j,n}(r), \tag{6}$$

whose coefficients

$$c_j = \sum_{r=0}^{n} u_r P_{j,n}(r) \tag{7}$$

may be regarded as known.

Replacing z_s by the values $Z_m(y_s)$, computed by means of the approximating polynomial (4), and using formulas (3) and (2), we arrive at a system of equations for the unknown coefficients b_k, having the form

$$\sum_{k=0}^{m} A_{jk} b_k = c_j \qquad (j = 0(1)m), \tag{8}$$

where

$$A_{jk} = \sum_{r=0}^{n} \sum_{s=0}^{n} K_{rs} P_{j,n}(r) P_{k,n}(s). \tag{9}$$

This system of equations should be solved numerically. It has lower order than the original system (2), and, with a proper choice of the degree m, is well conditioned.

To check the method described and to compare it with another method for solving ill-posed problems–the regularization method of A. N. Tikhonov–we solved a methodological example taken from [2]. In this example

$$K(x,y) = \frac{1}{\pi} \frac{1}{(x-y)^2 + 1}, \qquad z(y) = (1-y^2)^2, \qquad a = -1, \qquad b = 1. \tag{10}$$

In the solution the same intervals and approximations were used as in [2]. The only difference was that the function $u(x)$ was computed not analytically, but by formula (2). Owing to this, the accuracy of the solution of the integral equation (1) turns out not to be connected with the accuracy of computing the integral in the left-hand side of equation (1) by the rectangle formula. The inverse problem was then solved, the functions $u(x)$ and $z(y)$ being approximated by Chebyshev polynomials of the 4th, 6th, and 8th degrees. The results of the computations show that the functions $z(y)$ are almost completely recovered (for example, the polynomial $Z_4(y)$ and the function $z(y)$ coincide to 6 digits for all y).

Analogous computations were carried out for the functions

$$z(y) = \cos \lambda y, \qquad z(y) = \cos 3\lambda y, \qquad \lambda = 41\pi/180 = 0.71545. \tag{11}$$

The values of the absolute magnitudes of the differences $\Delta_m = |Z_m(y) - z(y)|$ are given in Table 1. All computations were performed on the “Minsk-2”machine with accuracy 10^{-7}. The computation time for one variant was less than one minute.

Comparing the results obtained with the results given in work (^{2}), we see that the proposed method is much more accurate than the method of A. N. Tikhonov (in A. N. Tikhonov’ s method one obtains a system of equations that is not sufficiently well conditioned; therefore the accuracy of the solution of problem (10) does not exceed two and a half digits). From a computational point of view, the proposed method is less cumbersome, since it requires solving a system of linear equations of substantially lower order.

Table 1

Values of the absolute magnitudes of the differences ($h = 2/41$)

y	$z(y) = (1-y^2)^2$ $\Delta_4 \cdot 10^6$	$z(y) = (1-y^2)^2$ $\Delta_6 \cdot 10^6$	$z(y) = (1-y^2)^2$ $\Delta_8 \cdot 10^6$	$z(y) = \cos\lambda y$ $\Delta_4 \cdot 10^6$	$z(y) = \cos\lambda y$ $\Delta_6 \cdot 10^6$	$z(y) = \cos\lambda y$ $\Delta_8 \cdot 10^5$	$z(y) = \cos 3\lambda y$ $\Delta_4 \cdot 10^5$	$z(y) = \cos 3\lambda y$ $\Delta_5 \cdot 10^6$	$z(y) = \cos 3\lambda y$ $\Delta_8 \cdot 10^5$
$0h$	0	1	1	2	0	2	179	34	2
$1h$	0	1	2	2	0	2	168	31	2
$2h$	0	1	2	2	0	1	137	21	1
$3h$	0	0	2	1	0	0	88	6	0
$4h$	0	0	1	0	0	0	26	10	0
$5h$	0	0	0	0	0	1	41	25	1
$6h$	0	1	1	1	0	2	107	36	2
$7h$	0	1	1	2	0	2	163	39	2
$8h$	0	2	1	3	0	2	201	34	2
$9h$	0	2	1	3	0	1	214	20	1
$10h$	0	2	0	3	0	0	196	1	0
$11h$	0	1	1	2	0	1	147	25	1
$12h$	0	0	2	1	0	2	65	45	3
$13h$	0	0	3	0	0	3	40	55	3
$14h$	0	2	2	2	0	3	158	50	3
$15h$	0	3	0	4	1	1	269	24	1
$16h$	0	4	3	5	1	0	343	18	1
$17h$	0	4	7	5	1	4	342	67	5
$18h$	0	2	8	3	0	6	217	95	7
$19h$	0	1	3	1	0	3	95	56	4
$20h$	0	8	15	10	2	8	668	126	10

The use of Chebyshev polynomials is expedient if the functions $u(x)$ and $z(y)$ are specified at discrete, equally spaced points of the interval (a, b), as when solving system (2). To solve the original integral equation (1), one can with the same success use an arbitrary complete system of orthonormal functions $f_k(x)$ satisfying, on the interval (a, b), the condition

$$\int_a^b f_j(x) f_k(x)\,dx = \delta_{jk} \qquad (j, k = 0, 1, 2, ...). \tag{12}$$

We shall approximate $z(y)$ by the sum

$$Z_m(y) = \sum_{k=0}^{m} b_k f_k(y) \tag{13}$$

with unknown coefficients b_k. The given function $u(x)$ is approximated by an analogous expression

$$U_m(x) = \sum_{j=0}^{m} c_j f_j(x). \tag{14}$$

If approximation is understood in the sense of the least quadratic deviation, then, as is known,

$$c_j = \int_a^b u(x) f_j(x)\,dx. \tag{15}$$

Proceeding as above, we again obtain, for the unknowns b_k, the system of equations (8), in which

$$A_{jk} = \int_a^b \int_a^b K(x, y) f_j(x) f_k(y)\,dx\,dy. \tag{16}$$

As $f_k(x)$ one may take, for example (for $a = -\pi,\ b = \pi$), the normalized trigonometric functions $\cos kx$ and $\sin kx$. Then the sums (13) and (14) correspond to Fourier series.

The ill-posedness of the problem under consideration manifests itself only in the fact that in formulas (4) and (13) the degree m cannot be taken arbitrarily large if the accuracy of the computations, the accuracy with which the function $u(x)$ is specified, and the number of intervals n are fixed. The minimum values of m can be estimated by using a priori information about the behavior of the sought function $z(y)$ and by investigating the accuracy with which the given function $u(x)$ is approximated by expressions (6) and (14). The final choice of m must be made in the course of the computations and checked by them. This last circumstance cannot be regarded as a shortcoming of the proposed method, since it is characteristic of the practical application of almost all numerical methods.

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Note: Figure translations are in progress. See original paper for figures.

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