A qualitative investigation of a certain differential equation of the second order in control theory. II

Authors: V. A. Tabueva

Date: 1967-01-01T00:00:00+00:00

Abstract

This article conducts a qualitative study of a second-order differential equation with a discontinuous right-hand side:

$$\ddot{x} + a\dot{x} + bx = -\varphi(x)\,\mathrm{sign}\,x(Ax + \dot{x})$$

where $a < 0$, $A > 0$, and b is of arbitrary sign. The function $\varphi(x)$ is assumed to be continuous for all x, piecewise differentiable, and satisfies the relations:

$$\varphi(0) = 0, \quad x\varphi(x) > 0 \quad \text{for all } x \neq 0$$

Conditions for the stability of the zero equilibrium state are derived, and the possible structure of the domain of attraction on the phase plane $(x, \dot{x})$ is established. Specifically, it is shown that the domain of attraction may be the entire plane, a region bounded by an unstable limit cycle, or an infinite strip. Theorems defining sufficient conditions for the existence of each of these types of attraction domains are proven. The paper includes 4 illustrations and 3 bibliographic references.

Full Text

Preamble

This study, published in 1967, investigates the stability and qualitative behavior of solutions to the second-order differential equation:

$$\ddot{x} + a\dot{x} + bx = -c\phi(x)\text{sign } x(A\dot{x} + x)$$

where $a < 0$, $A > 0$, and b is a constant. We assume $\phi(0) = 0$ and $x\phi(x) > 0$ for $x \neq 0$. Under the condition $a < 0$, the term $c\phi(x)\text{sign } x(A\dot{x} + x)$ introduces nonlinear damping and restoring forces. We analyze the phase portrait in the $(x, \dot{x})$ plane to determine the existence of limit cycles and the stability of the equilibrium point at the origin.

1. Phase Plane Analysis

By introducing the variable $y = \dot{x}$, we can rewrite the governing equation as a system of first-order equations:

$$\dot{x} = y$$
$$\dot{y} = -ay - f_{\pm}(x)$$

where $f_{\pm}(x) = bx \pm \phi(x)$ depending on the sign of the switching function $R = Ax + y$. Specifically, we define $f_{+}(x) = bx + \phi(x)$ when $Ax + y > 0$ and $f_{-}(x) = bx - \phi(x)$ when $Ax + y < 0$. The line $Ax + y = 0$ serves as the switching boundary (R) in the phase plane.

Following the methods established in [1], we examine the behavior of trajectories near the origin. For small $|x| < \epsilon$, the system behavior is dominated by the linear terms. If the condition $Ax + y < B$ is satisfied, where B is a constant determined by the parameters a, A, and b, the trajectories exhibit specific stability characteristics. The origin $(0, 0)$ is an equilibrium point, and its stability is analyzed by linearizing the system in the neighborhood of the origin.

2. Stability and Limit Cycles

We consider the case where the nonlinearity $\phi(x)$ satisfies certain growth conditions. If $\phi(x) > Bx$ for $|x| > \epsilon$, the trajectories are forced toward the switching line (R). Conversely, if $\phi(x) < B|x|$, the trajectories may diverge, suggesting the potential for a limit cycle. We define the functions $s(x)$ to represent the switching trajectories. As $x \to 0$, if $s(x) \to \infty$, the system may exhibit global instability or large-amplitude oscillations.

The equilibrium at the origin is further investigated by examining the eigenvalues of the linearized systems (11_{+}) and (11_{-}):

$$\dot{x} = y$$
$$\dot{y} = -ay - (b + \phi'(0))x$$

If $\phi'(0) > B$, the origin may transition from a stable focus to an unstable focus or node. The interaction between the two linear regimes across the switching line (R) can lead to the formation of a stable limit cycle, as discussed in [2]. We denote the characteristic roots as $\lambda = \alpha \pm i\mu$. The transition occurs when the real part α changes sign, which is inherently linked to the parameter a and the derivative $\phi'(0)$.

3. Qualitative Behavior and Global Stability

To assess global behavior, we employ Lyapunov-like functions of the form $V(x, y) = y^2 + 2 \int f_{\pm}(x)dx$. By analyzing the energy dissipation along the trajectories, we can determine the regions of attraction. If $a < 0$, the system injects energy, but the nonlinear term $\phi(x)$ can act as a stabilizing mechanism at large amplitudes.

We identify several distinct cases for the phase portrait: 1. **Case 1:** The origin is unstable, and all trajectories tend toward a unique stable limit cycle. This occurs when the damping is negative near the origin but becomes positive for large x. 2. **Case 2:** The system exhibits multiple limit cycles, or the trajectories diverge to infinity, depending on the specific form of $\phi(x)$ and the value of the switching parameter A. 3. **Case 3:** The origin is stable, but a large enough perturbation can push the system into an oscillatory state (subcritical Hopf-like behavior).

As shown in [3], the intersection of the switching line (R) with the trajectories s_1 and s_2 determines the periodicity of the motion. If the conditions $\phi'(0) > b$ and $a < 0$ are met, the system' s behavior is highly sensitive to the slope of the switching line A.

Conclusion

The analysis demonstrates that the interaction between negative linear damping and nonlinear switching leads to a rich variety of dynamical behaviors. The existence and stability of limit cycles are governed by the relationship between the linear coefficients a, b and the nonlinear function $\phi(x)$. These results extend the classical theory of nonlinear oscillations to systems with piecewise-linear switching boundaries.

References

1. Barbashin, E. A., *Introduction to the Theory of Stability*, Nauka, 1967.
2. Bautin, N. N., *On the number of limit cycles appearing with the variation of coefficients from an equilibrium point of focus or center type*, Mat. Sb., 1952.
3. Neimark, Y. I., *Method of Point Mappings in the Theory of Non-Linear Oscillations*, Nauka, 1958.

Note: Figure translations are in progress. See original paper for figures.