ON THE \(\varepsilon\)-ENTROPY OF ARZELÀ COMPACTA OF FUNCTIONS DEFINED ON CLOSED SETS OF POSITIVE LEBESGUE MEASURE

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 513.881

MATHEMATICS

A. F. TIMAN

ON THE ε-ENTROPY OF ARZELÀ COMPACTA OF FUNCTIONS DEFINED ON CLOSED SETS OF POSITIVE LEBESGUE MEASURE

(Presented by Academician A. N. Kolmogorov on 27 I 1967)

Let A be an arbitrary bounded closed set of n-dimensional Euclidean space R_n. Any nondecreasing continuous and subadditive function $\omega(t)$, defined on the half-axis $0 \le t < \infty$, determines the class $D_\omega^A(C)$ of all real functions $f(x) = f(x_1, \dots, x_n)$, given at the points $x \in A$, which satisfy the condition

$$|f(x') - f(x'')| \le \omega(|x' - x''|) \qquad (x', x'' \in A)$$

and the condition

$$|f(x)| \le C \qquad (x \in A).$$

The equality $\lim_{t\to 0} \omega(t) = 0$, by Arzelà' s theorem, is equivalent to the compactness of $D_\omega^A(C)$ in the uniform metric

$$\rho(f, \varphi) = \max_{x \in A} |f(x) - \varphi(x)|,$$

and the rate at which $\omega(t)$ decreases to zero as $t \to 0$ in general characterizes the measure of massiveness of the compactum $D_\omega^A(C)$ in this theorem.

In the well-known work of A. N. Kolmogorov ([1]), in connection with the problem of superpositions of functions having prescribed smoothness (see ([2])), and in the spirit of the ideas of information theory, as a measure of the massiveness of a compactum Q in a metric space R, there were introduced its absolute ε-entropy $H_\varepsilon(Q)$, the ε-entropy $H_\varepsilon^R(Q)$ of the set Q relative to R, and the ε-capacity $\mathcal{E}_\varepsilon(Q)$ (see ([3]), § 1). In the same article ([1]) the problem was posed of investigating the growth of these characteristics as $\varepsilon \to 0$ for various compacta Q occurring in analysis; the inequality

$$\mathcal{E}_{\varepsilon 2}(Q) \leq H_\varepsilon(Q) \leq H_\varepsilon^R(Q) \leq \mathcal{E}_\varepsilon(Q),$$

was given, its role in such an investigation was shown, and exact orders of growth of the ε-entropy (ε-capacity) were obtained for certain important cases.

One of the first results of the corresponding table of A. N. Kolmogorov (see ([1]), estimate III, 2) consists in the following: if A is some n-dimensional cube in R_n, and $\omega(t) = t^\alpha$ $(0 < \alpha \leq 1)$, then the following relation of order holds*:

$$\mathcal{E}_\varepsilon\{D_\omega^A(C)\} \asymp H_\varepsilon\{D_\omega^A(C)\} \asymp \varepsilon^{-n/\alpha} \qquad (\omega(t) = t^\alpha). \tag{2}$$

Further investigations, connected with the attempt to generalize this result to more massive functional compacta $D_\omega^A(C)$ arising when one considers Arzelà characteristics $\omega(t)$ that decrease slowly to zero, or other compact-in-themselves sets A (see ([4]), § 17; ([3]), § 9; ([5]), theorem 8; ([11]), pp. 3, 27), even for the simplest compacta $A \subset R_n$ (an interval, a rectangle, a parallelepiped), left open

* Here and below we use the notation adopted by N. Bourbaki in ([9]), Chap. 5, for comparing infinitely small and infinitely large quantities.

the question of the exact order of growth of the ε-entropy $H_\varepsilon\{D_\omega^A(C)\}$ and the ε-capacity $\mathfrak{E}_\varepsilon\{D_\omega^A(C)\}$ in these cases, reducing only to estimates of the form (see ([4]), § 17)

$$2^{H_{2\omega^{-1}(2\varepsilon)}(A)} \preccurlyeq \mathfrak{E}_{2\varepsilon}\{D_\omega^A(C)\} \preccurlyeq H_\varepsilon\{D_\omega^A(C)\} \preccurlyeq 2^{H_{1/2\omega^{-1}(\varepsilon/2)}(A)} \tag{3}$$

under the assumption that A is connected, and to estimates of the form (see ([3]), § 9, p. 77)

$$2^{H_{2\omega^{-1}(2\varepsilon)}(A)} \preccurlyeq \mathfrak{E}_{2\varepsilon}\{D_\omega^A(C)\} \preccurlyeq H_\varepsilon\{D_\omega^A(C)\} \preccurlyeq 2^{H_{1/2\omega^{-1}(\varepsilon/2)}(A)} \log\frac{1}{\varepsilon} \tag{4}$$

without this assumption.

The estimate (3), due to A. G. Vitushkin ([4]), makes it possible to obtain the order of growth of the ε-entropy $H_\varepsilon\{D_\omega^A(C)\}$ for connected compact sets $A \subset R_n$, generally speaking, only when $\omega^{-1}(2\varepsilon) \asymp \omega^{-1}(\varepsilon)$. Even in the case where A is a finite interval of the real axis, it is not difficult to give examples showing that without this additional restriction on the massiveness of $D_\omega^A(C)$ the extreme terms of inequality (3) may turn out to be infinitely large quantities of different orders.

In connection with the problem under consideration, the author ([7]) established the following

Theorem 1. *If a connected compact set A in a metric space R satisfies the condition*

$$2^{H_\varepsilon(A)} \asymp 2^{H_\tau(A)} \tag{5}$$

for $\varepsilon \asymp \tau$, *then always*

$$0 < \varliminf_{\varepsilon\to 0} H_\varepsilon\{D^A_\omega(C)\} \cdot 2^{-H_{\omega^{-1}(2\varepsilon)}(A)} < \infty, \tag{6}$$

$$\varlimsup_{\varepsilon\to 0} H_\varepsilon\{D^A_\omega(C)\} \cdot 2^{-H_{\omega^{-1}(2\varepsilon)}(A)} < \infty, \tag{7}$$

where

$$\omega(t_1) + \omega(t_2) \le 2\omega\left(\frac{t_1 + t_2}{2}\right). \tag{8}$$

In particular, if the closed bounded set A belonging to the space R_n is connected and

$$H_\varepsilon(A) = n \log_2 1/\varepsilon + O(1) \tag{9}$$

(for example, if A is an n-dimensional parallelepiped), then, when condition (8) is satisfied, the relation of exact order is always valid (see ([6]))

$$\mathfrak{E}_{2\varepsilon}\{D^A_\omega(C)\} \asymp H_\varepsilon\{D^A_\omega(C)\} \asymp \left\{\frac{1}{\omega^{-1}(2\varepsilon)}\right\}^n, \tag{10}$$

showing that the order in the right-hand estimate (3) is crude if $\omega(t)$ tends to zero sufficiently slowly.

The last result, giving exact orders of the ε-entropy and ε-capacity of arbitrarily massive Arzelà compacta of real functions $f(x_1, \dots, x_n)$ of n real variables and revealing an essential difference in their growth as $\varepsilon \to 0$, naturally leads to the question of what the sets $A \subset R_n$ are for which (10) always holds. For such sufficiently massive compacta $Q = D^A_\omega(C)$, in the orders of growth of the corresponding terms of inequality (1) as $\varepsilon \to 0$ there appear arbitrarily large gaps, whose absence in other cases plays the decisive role in obtaining the known estimates (see ([3])).

Theorem 2. *For relation (10) to hold it is necessary, and when condition (8) is satisfied it is sufficient, that the compact set* $A \subset R_n$ *have positive* n*-dimensional Lebesgue measure.*

The proof of this assertion is based on Theorem 1, the consideration of Hausdorff p-measures ([10]), and the extension of functions with preservation of their modulus of continuity and maximum modulus.

Theorem 2 shows that the requirement that the set $A \subset R_n$ be connected is not dictated by the nature of the question. In particular, the result of A. N. Kolmogorov ([2]), for any natural n, remains valid for all closed sets $A \subset R_n$ having positive n-dimensional Lebesgue measure, and only for them. For $n = 1$ this conclusion also follows from the estimate recently obtained by Vosburg ([8]) for one-dimensional compacta A and $\omega(t) = t^\alpha$ $(0 < \alpha \le 1)$,

$$H_\varepsilon\{D^A_{t^\alpha}(1)\} \preccurlyeq N_\delta(A)\log\{2\varepsilon^{-1}[N_\delta(A)]^{-\alpha}\} + \log\frac{1}{\varepsilon},$$

where $N_\delta(A) = 2^{H_\delta(A)}$, $\delta = \varepsilon^{1/\alpha}$, if one uses the particular case $n = 1$ (see ([8])) of the following general proposition.

Theorem 3. *For any natural n, the asymptotic equality (9) for a bounded closed set $A \subset R_n$ holds if and only if the n-dimensional Lebesgue measure of A is positive.*

The last proposition complements the first of the estimates given in the above-mentioned table of A. N. Kolmogorov ([1]) (see ([3]), Theorem VIII). In the case when the Lebesgue measure of the boundary of the set $A \subset R_n$ is zero (the set A has "volume"), excluding, for example, everywhere disconnected sets of positive measure, the asymptotic estimate of $H_\varepsilon(A)$ and $C_\varepsilon(A)$ as $\varepsilon \to 0$ is known in a somewhat more precise form than (9) (see ([3]), § 4, Theorem IX).

Dnepropetrovsk Chemical-Technological
Institute

Received
27 I 1967

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Note: Figure translations are in progress. See original paper for figures.