SOLUTION OF THE GENERALIZED HARDY-LITTLEWOOD PROBLEM IN SECTORS

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Abstract

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SOLUTION OF THE GENERALIZED HARDY-LITTLEWOOD PROBLEM IN SECTORS

(Presented by Academician Yu. V. Linnik on 19 IX 1966)

In paper (1), proceeding from ergodic considerations, B. M. Bredikhin and Yu. V. Linnik obtained an asymptotic formula for $Q_\Delta(n)$—the number of solutions (p, x, y) of the equation

$$p + x^2 + y^2 = n \quad (n \text{ natural}) \tag{1}$$

in primes p and integers x, y, for which the integer point (x, y) falls into the sector Δ of prescribed aperture ψ.

The solution of the generalized Hardy-Littlewood equation

$$p + \varphi(x, y) = n \tag{2}$$

in sectors was made difficult by the fact that the quadratic field corresponding to the discriminant of the quadratic form is, generally speaking, non-principal.

In paper (2) B. M. Bredikhin and Yu. V. Linnik obtained an asymptotic formula for $Q(n)$—the number of all solutions of (2) under natural restrictions. In doing so they singled out "good" numbers, representable by the totality of all inequivalent forms of the same negative discriminant as that of $\varphi(x, y)$ from the given genus to which $\varphi(x, y)$ belongs, and also a "good" genus—the principal genus. It turned out that, for solving problem (2) in sectors, these two choices are insufficient: one also needs the notion of a "good" class. This turned out to be the principal class.

Let us note that the result of paper (1) can also be obtained by replacing the ergodic considerations with a consideration of trigonometric sums over Hecke characters of the first kind

$$S_m^{(k)} = \sum_{x^2+y^2=m} e^{ki \arg(x+iy)}, \tag{3}$$

where k is an integer; m is natural; x, y run through all integers satisfying the equation $x^2 + y^2 = m$.

In this note we shall give a brief outline of the proof of the following theorem, which is the solution of the generalized Hardy-Littlewood problem in sectors.

Theorem. Let $Q_\Delta(n)$ be the number of solutions (p, x, y) of equation (2) in primes p and integers x, y, for which

$$0 \le \psi_1 \le \arg(x + iy) \le \psi_2 \le 2\pi,$$

where ψ_1 and ψ_2 are fixed real numbers, and let $Q(n)$ be the number of all solutions (p, x, y) of equation (2) in primes p and integers x, y.

Suppose that $\varphi(x, y) = Ax^2 + Bxy + Cy^2$ is a primitive positive quadratic form with negative discriminant $-d = B^2 - 4AC$, free of odd squares, and, moreover, either $-d = -4D$, $-D \equiv 2, 3 \pmod 4$, or $-d = -D$, $-D \equiv 1 \pmod 4$. Suppose that the interval $[\psi_1, \psi_2]$ contains neither the points $\pi/2, 3\pi/2$ nor the points x for which $\operatorname{tg} x = -2A/B$ when $B \ne 0$.

Then, as $n \to \infty$,

$$Q_\Delta(n) = \frac{Q(n)}{2\pi} \left[\operatorname{arctg} \frac{\sqrt{d}\ \operatorname{tg}\psi_2}{2A + B \operatorname{tg}\psi_2} - \operatorname{arctg} \frac{\sqrt{d}\ \operatorname{tg}\psi_1}{2A + B \operatorname{tg}\psi_1} \right] + O\left(\frac{Q(n)}{(\log\log n)^{1/2-\eta}} \right), \tag{4}$$

where η is fixed, $0 < \eta < 1/2$ (Δ is the sector of aperture $\psi_2 - \psi_1$).

Let us consider the scheme of the proof of this theorem. Divide the circle with center at the origin and radius $\sqrt{n}$ ($n > n_0$) into K equal sectors δ_i ($i = 1, \dots, K$) with apertures $\varepsilon = \varepsilon_i - \varepsilon_{i-1}$, where $\varepsilon_0 = 0$, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 2\varepsilon, \dots, \varepsilon_K = K\varepsilon$, $\varepsilon = 2\pi/K$. Take $K = 2^{2R-1}$ and

$$\frac{1}{4}(\log\log n)^{1-\eta} < K < (\log\log n)^{1-\eta},$$

where η is fixed, $0 < \eta < 1$.

It is clear that

$$Q_\Delta(n) = \sum_{m=n-p} \sum_{\substack{\varphi(x,y)=m \\ (x,y)\in\Delta}} 1. \tag{5}$$

Further,

$$Q(n) = \sum_{\varphi(x,y)=m,\ p+m=n} \sum 1 = \sum_{p+m=n} r(m),$$

where $r(m)$ is the number of integral points (x, y) on the ellipse $\varphi(x, y) = m$.

Thus,

$$Q(n) = \sum_{p+m=n} r(m). \tag{6}$$

We divide the set of numbers m satisfying (6) into two classes A and B. In class A we include those m $(1 \leq m \leq n)$ that satisfy the following conditions.

For each $i = 1, 2, \dots, [K/w]$, where w is the number of roots of unity belonging to the field $R(\sqrt{-d})$, obtained by adjoining to the field of rational numbers R the number $\sqrt{-d}$, the canonical decomposition of m contains at least one prime number p_i to exactly the first power such that $(-d/p_i) = +1$ ($(-d/p_i)$ is the Legendre symbol), and p_i can be decomposed into prime principal ideals of the field $R(\sqrt{-d})$, $p_i = \rho_i \bar{\rho}_i$, with $\rho_i \in \delta_i$ ($\rho_i \in \delta_i$ means that the number ρ_i corresponding to the principal ideal falls into the sector δ_i), where $\bar{\rho}_i$ is the ideal conjugate to ρ_i.

In class B we include all the remaining m $(1 \leq m \leq n)$. Estimating the sum

$$\sum_{m=n-p,\ m\in B} r(m)$$

from above by sieve methods (3–5), we obtain

$$\sum_{m=n-p,\ m\in B} r(m) = O\left(K \frac{n(\log\log n)^3}{\log n \cdot (\log n)^{w/Kh}}\right), \tag{7}$$

where h is the number of ideal classes of the field $R(\sqrt{-d})$.

Further, with the aid of Hecke characters of the second kind, the following lemma is proved:

Lemma. Let $\varphi(x, y)$ be a form satisfying the conditions of the theorem; let $r(m)$ be the number of all integral points on the ellipse

$$\varphi(x, y) = m; \tag{8}$$

let $T_\Delta(\psi_1, \psi_2, m)$ be the number of integral points on the ellipse (8) that fall inside the sector Δ with aperture $\psi_2 - \psi_1$, where the segment $[\psi_1, \psi_2]$ satisfies the conditions of the theorem. Suppose that in the decomposition of m into prime factors there enters at least one prime number p_i satisfying $(-d/p_i) = +1$, exactly to the first power, and such that for each $i = 0, 1, 2, \dots, R-2$,

$$p_i = \rho_i \bar{\rho}_i,$$

where ρ_i is a prime principal ideal of the field $R(\sqrt{-d})$ (the prime number corresponding to this principal ideal will be denoted by the same letter ρ_i), ρ_i– conjugate to ρ_i and

$$2^i\varphi \leqslant \arg \rho_i \leqslant 2^i\varphi + \varepsilon, \qquad i = 0, 1, \dots, R-2, \quad \varphi = \frac{2\pi}{2^R}, \quad \varepsilon = \frac{2\pi}{2^{2R-1}}. \tag{9}$$

Then

$$T_\Delta(\psi_1, \psi_2, m) = \frac{r(m)}{2\pi}\left[\operatorname{arc\,tg}\frac{\sqrt{d}\,\operatorname{tg}\psi_2}{2A + B\operatorname{tg}\psi_2} - \operatorname{arc\,tg}\frac{\sqrt{d}\,\operatorname{tg}\psi_1}{2A + B\operatorname{tg}\psi_1}\right] + O\left(\frac{r_\varphi(m)}{2^R}\right), \tag{10}$$

where $r_\varphi(m)$ is the number of representations of m by the system of all inequivalent h forms of discriminant $-d$.

By Theorem 1 of [2], together with (5), (6), (7), and (10), we obtain the proof of the theorem.

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