Regularity of a class of linear systems with almost periodic coefficients

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Abstract

The paper considers a class of systems of linear differential equations with almost-periodic coefficients. It is proved that the systems of the selected class are regular. Bibliography: 6 items.

Full Text

Preamble

In 1967, following the methodologies established in [1], we consider the differential equation:

$$\dot{X} = \mu(P_0 + \mu P_1 + \mu^2 P_2 + \dots)X \tag{0.1}$$

where $P_k(t)$ are matrices and μ is a small parameter. We seek a transformation of the form:

$$X(t,\mu) = Z(t,\mu)\exp\left\{\int A(t,\mu)dt\right\} \tag{0.2}$$

where the matrices A and Z are represented by the formal power series:

$$A(t,\mu) = \sum \mu^k A_k(t) \tag{0.3}$$

$$Z(t,\mu) = E + \sum \mu^k Z_k(t) \tag{0.4}$$

Substituting (0.2) into (0.1) and equating coefficients of like powers of μ, we obtain the recurrence relations for $Z_k(t)$ and $A_k(t)$:

$$\begin{aligned} L_{P_0} Z_n &= P_n + P_{n-1}Z_1 + \dots + P_1 Z_{n-1} - A_n - (Z_1 A_{n-1} + \dots + Z_{n-1}A_1) \\ L_{P_0} Z_n &= \dot{Z}_n - P_0 Z_n + Z_n P_0 \end{aligned} \tag{0.5}$$

The matrices $A_k(t)$ are chosen to satisfy specific structural requirements, typically being diagonal or having a simplified block structure, while $Z_k(t)$ are determined to ensure the formal consistency of the expansion. If the conditions of the fundamental theorems in [1] are met, the series (0.3) and (0.4) provide an asymptotic representation of the solution to (0.1).

§ 1. Preliminary Estimates and Function Spaces

Let Φ denote a space of functions $f(t)$ such that the linear operator $L_a(t)y = \dot{y} + a(t)y = f(t)$ possesses a bounded solution in the same space. We consider the equation:

$$\dot{y} + a(t)y = f(t) \tag{1.1}$$

where $a(t) \in \Phi$. If $Z(t)$ is an oscillatory or almost-periodic function, we denote its mean value as $\bar{Z}$. The solution to (1.1) can be expressed via the integral operator:

$$y(t) = \exp\left\{-\int a(x)dx\right\} \int \exp\left\{\int a(x)dx\right\} f(x)dx \tag{1.2}$$

Under the condition $\mathrm{Re}\ a > 0$, and assuming $a(t), f(t) \in \Phi$, the stability of the solution is guaranteed. Following the methods of [2] and [6], we establish bounds for the norm of the solution $\|y\|$. Specifically, if $\mathrm{Re}\ a > \sup |a(t)|$, then the integral (1.2) converges and satisfies:

$$|y(t)| \leq \frac{1}{\inf |\mathrm{Re}\ a(t)|} \sup |f(t)|$$

Furthermore, if $a(t)$ is decomposed as $a(t) = \bar{a} + \phi(t)$, where $\phi(t)$ represents the fluctuating component, we can refine these estimates using the properties of the exponential growth of the fundamental solution. As shown in [3], for sufficiently large N, the transformation $y(t) = \exp\{-\int s_k(x)dx\}u(t)$ allows us to reduce the problem to a form where the operator L_a is more easily inverted.

§ 2. Asymptotic Transformations of the System

We now extend these results to the matrix case. Consider the operator $L_P(t)Z = \dot{Z} - P(t)Z + ZP(t) = F(t)$. Let $\lambda_j(P)$ denote the eigenvalues of the matrix P. We assume that the eigenvalues $\lambda_k(P_0)$ are distinct and satisfy the condition $\mathrm{Re}\ \lambda_j(P_0) \neq \mathrm{Re}\ \lambda_k(P_0)$ for $j \neq k$. Under these assumptions, the operator L_{P_0} is invertible in the space of matrices with entries in Φ.

The transformation (0.2) leads to the system of equations (2.6) for the terms of the series. By applying the results of § 1 to each component, we demonstrate that the formal series for $Z(t, \mu)$ and $A(t, \mu)$ are well-defined. Specifically, the matrices $A_k(t)$ are chosen as:

$$A_k(t) = \mathrm{diag}\{P_k(t) + P_{k-1}Z_1 + \cdots + P_1 Z_{k-1}\}$$

This choice ensures that the remaining terms in (2.6) can be solved for $Z_k(t)$ such that $Z_k \in \Phi$. As established in [1] and [4], the convergence of these series in the sense of asymptotic expansions holds for $0 < \mu < R$, where R is a radius of convergence determined by the norms of the operators $L_{P_0}^{-1}$ and the smoothness of the matrices $P_k(t)$.

The system (0.1) can be transformed into a diagonal or block-diagonal form:

$$\dot{y} = \left(\sum_{k=0}^{n} \mu^k A_k(t) \right) y + \mu^{n+1} \Phi_{n+1}(t, \mu) y \tag{2.13}$$

where the remainder term Φ_{n+1} is bounded. This reduction allows for the application of standard stability criteria and the construction of approximate solutions with a high degree of precision.

§ 3. Example and Applications

To illustrate the method, consider the case where P_0 is a constant matrix and $P_1(t)$ is a periodic matrix of the form:

$$P_1(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \tag{3.1}$$

Using the recurrence relations (0.5), we calculate the first-order correction $Z_1(t)$ and the effective matrix A_1. The integration of the diagonal components yields the secular terms in the phase of the solution. If the mean value of the diagonal elements of P_1 is non-zero, it results in a shift of the characteristic exponents. The results obtained here are consistent with the general theory of linear systems with periodic coefficients as discussed in [5].

References

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Note: Figure translations are in progress. See original paper for figures.

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