On the definition of the capture regions in the case of a system of linear differential equations with periodic coefficients

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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DETERMINATION OF CAPTURE REGIONS FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

Introduction

Consider a system of equations:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \phi_{ij}(t+\tau)x_j, \quad i = 1, 2, \dots, n \tag{1}$$

where ϕ_{ij} are real continuous periodic functions with period ω, and τ is a parameter. The solution to this system can be written in the form:

$$x_i(t, \tau, x_{1,0}, x_{2,0}, \dots, x_{n,0}) = \sum_{j=1}^{n} u_{ij}(t, \tau)x_{j,0} \tag{2}$$

where the quantities u_{ij} are defined as:

$$u_{ij}(t, \tau) = x_i(t, \tau, 0, \dots, 1, \dots, 0)$$

We define the **capture region** with respect to the variables $x_1, x_2, \dots, x_n$ as the set of all points in n-dimensional space such that, for any $t > 0$, the formulas (2) yield a point in $(p+1)$-dimensional space lying within the $(p+1)$-dimensional parallelepiped:

$$|x_j| \leq a_j, \quad j = i, i+1, \dots, i+p$$

where a_j are given positive constants.

In the present work, we calculate quantities that generally provide a lower bound for the capture regions for the cases $n = 2$ and $n = 3$. Furthermore, a non-trivial example is presented where the calculated value, under certain known conditions, yields the exact magnitude of the capture region.

§ 1. INTEGRALS OF MOTION OF A SYSTEM OF TWO EQUATIONS

Consider a system of equations given by:

$$\frac{dx_i}{dt} = \sum_{j=1}^{2} \phi_{ij}(t+\tau)x_j, \quad i = 1, 2 \tag{1.1}$$

where ϕ_{ij} are continuous, periodic real-valued functions with a period of ω, and τ is a parameter. It is well known [**?**] that in cases where the characteristic roots of (1.1) are distinct, the matrix of linearly independent solutions for this system can be expressed as follows:

$$W(t,\tau) = \exp\left[\int_0^t \Phi(\xi+\tau)\,d\xi\right] \tag{1.2}$$

The solution can be represented using the following transformation:

$$\Psi(z) = [f(z)]\Phi(z) + [g(z)]\Phi^*(z) \tag{1.3}$$

where the square brackets denote a diagonal matrix; f, g are complex-valued functions, and the asterisk denotes complex conjugation. Here, $\Psi(z)$ represents a periodic solution to the nonlinear equation:

$$\mathcal{L}\Psi = 0 \tag{1.5}$$

We proceed by separating the equation into its real and imaginary parts:

$$w_{21} = u_{21} + iv_{21} \tag{1.6}$$

and we shall assume that $k > 0$. In place of the function u, we introduce another function v according to the relation:

$$v = \frac{u}{\phi_{11} - \phi_{22}} \tag{1.7}$$

By substituting (1.6) into (1.5) and subsequently separating the real and imaginary parts, we find that the function v satisfies the equation:

$$v'' - \frac{\phi_{12}'}{\phi_{12}} v' + (\phi_{11} - \phi_{22}) v = 0 \tag{1.8}$$

and the function v is expressed in terms of u as follows: $u = v(\phi_{11} - \phi_{22})$ (1.9). Equation (1.8) generally possesses complex periodic solutions. However, we shall assume that the coefficients ϕ_{ij} are such that the function is real-valued. Let us introduce the notation:

$$\eta = \phi_{11} + \phi_{22}$$

The solution to (1.1) is written using (1.2) as follows:

$$x(t) = W^{-1}(z+t) W(z) x_0 \tag{1.12}$$

By applying (1.3) and the subsequent formulas, we find that (1.12) is equivalent to the equation:

$$s_1 + i s_2 = S_1(x,t) + i S_2(x,t) \tag{1.14}$$

By equating the moduli and phases of the complex quantities on the right and left sides of (1.14), we obtain:

$$s_1^2(x,z,t) + s_2^2(x,z,t) = \exp\left(\int_0^t (\phi_{11} + \phi_{22})\, d\xi \right) \tag{1.15}$$

$$\partial_t \sigma(X,Z,t) = \sigma(X,z) \tag{1.16}$$

where $\sigma(X,z) = \sigma(X,z,0)$ (1.18). In the following, we shall refer to (1.15) as the first integral and to (1.16) as the second integral of the system (1.1).

§ 2. STUDY OF THE FIRST INTEGRAL OF THE SYSTEM

We solve equations (1.13) with respect to t, and let $u_{\pm}(x,z,t) = v(z \pm t) \pm s_2(x,z,t)$:

$$u(x,z,t) = \Phi u'(z+t) + \Phi u'(x,z,t) \tag{2.1}$$

Let us assign specific values to z and t, and choose the components of the vector x such that the expression $s_1^2(x,z) + s_2^2(x,z)$ remains constant. Then the values of (x,z,t) will correspond to the coordinates of the points on the circle (1.15) in the plane. To find the maximum absolute values of these functions on this circle, it is necessary to solve the relative extremum problem (see [**?**]). As a result, for the maximum of $|u|$, we obtain:

$$|u(x,z,t)| = \exp\left(-\int_0^t (\phi_{11} + \phi_{22})\, d\zeta \right) \sqrt{v^2(z+t) + s^2(x,z)} \tag{2.2}$$

We shall now determine two regions of values for the quantity $Q = \sqrt{S_1^2(x,z) + S_2^2(x,z)}$ (2.4) such that the values from the first region, when substituted into (2.2), and those from the second region, when substituted into (2.3), yield $|x_i| < \epsilon$ $(i = 1, 2)$ (2.5) for all t in the interval $0 < t < \infty$ (2.6).

Let us denote by $M_1(z)$ the minimum of the function $v(z+t)\exp\{-\int_0^t \phi_1(\tau)d\tau\}$ and by $M_2(z)$ the minimum of the function related to the second component. The required intervals are then given by $(i = 1, 2)$ (2.10). The formula for the area of the region in the (x, z, t) plane is:

$$A_i(z,t) = \pi M_i^2(z) \exp\left\{\int_0^t \phi_i(\tau)d\tau\right\} \tag{2.11}$$

By setting $t = 0$ in (2.11), we obtain a value that provides a lower bound for the area of the capture region:

$$J_i(z,0) = \pi M_i^2(z) \tag{2.12}$$

The final capture region area is limited by:

$$K(z) = \min[V_1(z,0), V_2(z,0)] \tag{2.13}$$

Consider the special case [?]: $\phi_{11}(z) = \phi_{22}(z) = 0, \phi_{12}(z) = 1, \phi_{21}(z) = -g(z)$. The regions in the plane will be the interiors of the ellipses:

$$\frac{x_1^2}{v^2(z)} + \frac{(v(z)x_2 - v'(z)x_1)^2}{1} \le \epsilon^2 \tag{2.16}$$

§ 3. STUDY OF THE SECOND INTEGRAL OF SYSTEM (1.1)

We write the function $w(x, z, t)$ for the case (2.14):

$$w(x,z,t) = P_i \cos\theta(x_1, x_2, z) \tag{3.1}$$

where

$$\tan\theta(x_1, x_2, z) = -\frac{1}{v(z)} \frac{x_2 v(z) - x_1 v'(z)}{x_1} \tag{3.3}$$

Consider the case where $v^*(z) = c_0(1 - 2b\cos 2\pi z)$ (3.4). For simplicity, let $z = 0$, $b = 1/2$, and $c_0 = -1/2$. We obtain:

$$u(x,0,t) = p(x_1, x_2, 0)\sqrt{2 - \cos(2\pi t)}\sin(2\pi t + \sigma) \tag{3.11}$$

The function (3.11) is periodic. It can be deduced that the maximum of $|\chi(x, 0, t)| < 1$ over a single period. Consequently, there exist points on the ellipse (3.9) that satisfy the condition $|u(x, 0, t)| < a$ for all t. If the function (3.11) were not periodic—which occurs when ω/π is irrational—the capture region would coincide exactly with the region bounded by the ellipse (3.9).

§ 4. INTEGRALS OF MOTION FOR THE SYSTEM OF FOUR EQUATIONS

Let there be a system of equations given by:

$$\phi_k = \phi_k(z) \quad (k = 1, 2, 3, 4) \tag{4.1}$$

where $\phi_k(z)$ are periodic real-valued functions with period 1. The matrix of linearly independent solutions is:

$$W(z) = \exp\left[\int \Phi(z, \xi)\, d\xi\right]; \quad \Phi = (\phi_{ik}) \tag{4.2}$$

The elements of the matrix are periodic solutions of the nonlinear system:

$$\dot{\Phi}_{ik} = \Phi_{ik} \tag{4.4}$$

We separate the real and imaginary parts: $\Phi_{ik} = u_{ik} + iv_{ik}$ (4.8). We assume the determinant $\Delta \neq 0$ (4.9). We transform to the variables s_1, s_2, s_3, s_4 (4.10). This leads to the first integrals:

$$|s|^2 + |\sigma|^2 = Q^2 \exp\left(2 \int \gamma\, d\xi\right) \tag{4.22}$$

$$sf^* + s^* f = Q \exp\left(2 \int \gamma\, d\xi\right) \tag{4.23}$$

and the second integrals:

$$\phi(x, z, t) = \int \omega\, d\xi + \Phi(x, z, 0) \tag{4.24}$$

$$\phi(x, z, t) = \Phi(x, z, 0) \tag{4.25}$$

§ 5. INVESTIGATION OF THE FIRST INTEGRALS OF SYSTEM (4.1)

We solve equations (4.10) for $s_\ell + a_\ell$:

$$s_\ell + a_\ell = \frac{j_\ell + a_\ell}{(z + t)s_\ell} \tag{5.1}$$

We seek the maximum value $m_i(x, z)$ subject to constraints (4.22) and (4.23):

$$m_i(x, z) = Q_1 \sqrt{a_1^2(z + t) + a_1^2} + Q_2 \sqrt{a_2^2(z + t) + a_2^2} \tag{5.2}$$

The region where every point satisfies inequality (2.5) for all t is bounded by the coordinate axes and a curve

DIFFERENTIAL EQUATIONS

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1. Researching pendulum systems (simple pendulum, double pendulum, etc.), as well as the dynamics of electromechanical systems, integrated circuits, synchronization of phase circuits, adjusting frequency, leads to the necessity of considering systems of differential equations of the form

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CLASSIFICATION OF TRAJECTORIES OF A DYNAMICAL SYSTEM WITH CYLINDRICAL PHASE SPACE

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where the variables $\varphi_1, \ldots, \varphi_m$ are angular variables (phase) coordinates and functions Φ_i, X_j are periodic functions (with period 2π) of these coordinates, variables $x_1, \ldots, x_n$ are Euclidean coordinates.

Without loss of generality, it can be assumed that is points of the form

$$\frac{dq_i}{dt} = \Phi_i(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n) \quad (i = 1, \ldots, m),$$
$$\frac{dx_j}{dt} = X_j(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n) \quad (j = 1, \ldots, m), \tag{1}$$

Without loss of generality, it can be assumed that the period of all angular coordinates is the same and equals 2π. This means that the physical state of the system under consideration described by points of the form $(\varphi_1 + 2k_1\pi, \ldots, \varphi_m + 2k_m\pi, x_1, \ldots, x_\eta)$, where $k_1, \ldots, k_m$ are integers, is identically. Identifying all points of the indicated form, we obtain a cylindrical phase $\mathbb{R}(\varphi_1, \ldots, \varphi_m)$. This space as the "topological product of an m-dimensional torus and an n-aindian space of variables $x_1, \ldots, x_n$.

The Euclidean space $\mathbb{R}$ of variables $\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n$ will be the covering space for the cylindrical space $R(\varphi_1, \ldots, \varphi_m)$ (see, for example, [2] and [3]).

Cylindrical space $R(\varphi_1, \ldots, \varphi_m)$ can be obtained from space R if the latter is cut along the surfaces $\varphi_i = -\pi$, $\varphi_i = \pi$ $(i = 1, \ldots, m)$ and gluing is performed.

Cylindrical space $R(\varphi_1, \ldots, \varphi_m)$ can be obtained from space R if the latter is cut along the surfaces $\varphi_i = -\pi$, $\varphi_i = \pi \ldots, \varphi_i = \pi$ $(i = 1, \ldots, m)$ and gluing is performed of the resulting "strip" along the cut surfaces. Clearly, that such folding can be performed not for all coordinates $\varphi_1, \ldots, \varphi_m$, but only for a certain part of these coordinates, for example, for coordinates $\varphi_1, \ldots, \varphi_j$; the cylindri— cal space thus obtained will be denoted by the symbol $R(\varphi_1, \ldots, \varphi_m)$. Obviously, space $R(\varphi_1, \ldots, \varphi_n)$ is also the covering for the space $R(\varphi_1, \ldots, \varphi_m)$.

Assuming the fulfillment of some conditions ensuring the existence and continuability of the solutions of the system of equations (1), we obtain on the phase space $R(\varphi_1, \ldots, \varphi_m)$ a dynamical system. This dynamical system induces in any covering space

Figure 1: Figure 1

reduced with the help of continuous deformation into another. The number of different independent classes of closed paths is called the connectivity order of the manifold. The connectivity order of the space $R(\varphi_1, \dots, \varphi_m)$ is equal to $m+1$, thus, the maximum number of homotopically independent limit cycles in the space $R(\varphi_1, \dots, \varphi_m)$ is equal to $m+1$.

Such a maximal system can be represented, for example, by a system consisting of a 0-cycle and φ_i-cycles, where $i = 1, \dots, m$.

Thus, the classification of limit cycles proposed by us provides a finer partition into classes in comparison with the classification based on the concept of homotopy, since all cycles of class, different from class (0) and classes (φ_i), $i = 1, \dots, m$, will be homotopically dependent on the above-mentioned cycles. Nevertheless, our classification, distinguished by its simplicity, allows us to accurately characterize the location of the limit cycle on the considered cylindrical phase space.

For example, on a two-dimensional torus $R(\varphi_1, \varphi_2)$ we have three homotopically independent classes of limit cycles: 0-cycles, φ_1-cycles, encompassing the torus along the meridian, φ_2-cycles, encompassing the torus along the parallel. Obviously, (φ_1, φ_2)-cycles will encompass the torus both along the meridian and along the parallel, i.e., despite the fact that they are derivatives of the above-mentioned cycles, they can be of independent interest in research.

3. As property A, any other property of the trajectories of the dynamic system can be taken. One of such properties can be, for example, the property of positive stability of trajectories according to Poisson ([5], p. 363).

Recall that a point p is called positively stable according to Poisson, if for any neighborhood U of this point one can specify a positive cımber T such that, during its movement along the trajectory over a time interval $t \geq T$, the point p falls again into the neighborhood U at learst once. Note that if at least one point of the trajectory is positively stable according to Poisson, then all other points of this trajectory will have the same property. Thus, we obtain the concept of a positively stable trajectory according to Poisson (P–stable trajectory).

Following definition 2, we say that a trajectory from the space $R(\varphi_1, \dots, \varphi_m)$ is a P-stable trajectory of class $(\varphi_r, \dots, \varphi_s)$, if the property of P-stability is preserved when unrolled along all owch coordinates φ_i, where i is distinct from $r, \dots, s$, and disappears upon further unrolling along as now umoɔpears thon a further unrolling along any of the coordinates $\varphi_r, \dots, \varphi_s$.

If we take the two-dimensional torus as the space $R(\varphi_1, \varphi_2)$, then the only P-stable trajectories on it, distinct from rest points and limit cycles, will be P-stable trajectories of class (φ_1, φ_2). In Indeed, the unfolding of the torus along any of the coordinates φ_1, φ_2 is a cylinder, and along both coordinates φ_1, φ_2 – a place. But neither on the plane, nor on the two-dimensional cylinder can there be P—stable trajectories distinct from rest points and limit cycles (in both rerrate according cycles. [5], p. 364). Thus, on the torus there exist only P-stable trajectories of class (φ_1, φ_2), distinct from singular points and limit cycles. The property of P-stability of these trajectories is lost upon unrolling along any of the coordinates φ_1, φ_2.

If, however, as the space $R(\varphi_1, \varphi_2)$ we take the topological product of the torus by the real line, i.e., we introduce into consideration a new coordinate x, which is not angular, then the unfolding of such a space

Figure 2: Figure 2

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Following definition 2, we say that a trajectory from the space $R(\varphi_1, \dots, \varphi_m)$ is a P-stable trajectory of class $(\varphi_r, \dots, \varphi_s)$, if the property of P-stability is preserved when unrolled along all owch coordinates φ_i, where i is distinct from $r, \dots, s$, and disappears upon further unrolling along as now umoyears thon a further unrolling along any of the coordinates $\varphi_r, \dots, \varphi_s$.

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If, however, as the space $R(\varphi_1, \varphi_2)$ we take the topological product of the torus by the real line, i.e., we introduce into consideration a new coordinate x, which is not angular, then the unfolding of such a space

Figure 3: Figure 2

. Let $M(x, z)$ denote the minimum of Q within the interval (2.7). The parametric equation of the boundary curve is:

$$Q_1 = Q_1(x, z), \quad Q_2 = Q_2(x, z) \tag{5.7}$$

The four-dimensional volume corresponding to the values of Q and $\bar{Q}$ is:

$$V = \frac{Q_s(z)}{\Phi} \int \Phi(Q, r) Q \, dQ \tag{5.16}$$

By setting $\xi = 0$ in (5.16), we obtain the volume:

$$Q_i(z) = \int_0^\infty \dots \int_0^\infty Q \, dQ \tag{5.17}$$

which serves as a lower bound for the measure of the capture region. In conclusion, I express my gratitude to A. D. Myshkis and Yu. S. Bogdanov for valuable discussions.

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Figures

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spaces will be three-dimensional cylinders, and the full unfolding — three-dimensional Euclidean space. Obviously, in this case it is not difficult to construct examples of *P*-stable trajectories, distinct from singular points and limit cycles, of any of the four possible classes.

4. Let us now formulate a criterion for the absence of *P*-stable trajectories.

Theorem 1. Let there exist in the space R a single-valued continuously differentiable scalar function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$, the derivative of which, taken by virtue of system (1), is sign-constant. Let the function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ be periodic (with period 2π) with respect to the coordinates $\varphi_r, \ldots, \varphi_s$. Then all *P*-stable trajectories of class $(\varphi_r, \ldots, \varphi_s)$ (as well as class (0) and class $(\varphi_p, \ldots, \varphi_q)$, where φ_p, $\varphi_p, \ldots, \varphi_q$—coordinates from the set $\varphi_r, \ldots, \varphi_s$) lie on the set $v = 0$.

Thus, if the set $v = 0$ does not contain whole trajectories, then — *P*-stable trajectories of the classes indicated in the theorem are absent.

For the proof of the theorem, let us consider the space $R(\varphi_r, \ldots, \varphi_s)$, which is obtained from the space R by folding along the coordinates $\varphi_r, \ldots, \varphi_s$. It is easy to see that the function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ is single-valued and continuous, the space $R(\varphi_r, \ldots, \varphi_s)$, since the values of this function at corresponding points upon folding will be identical.

Let us assume now for definiteness that the function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ is non-positive in the space R, i.e., satisfies everywhere the inequality $v \leqslant 0$.

Let us assume further that there exists a *P*-stable trajectory of class $(\varphi_r, \ldots, \varphi_s)$. By definition, this trajectory is *P*-stable . . . in the space $R(\varphi_r, \ldots, \varphi_s)$ and loses this property upon unfolding along any angular coordinate $\varphi_r, \ldots, \varphi_s$.

Let us assume that on our trajectory there is at least one point q, at which v is distinct from zero, i.e., is negative. By definition of *P*-stability, for any positive number T it is possible to indicate a number $t > T$ such that after a time interval t the point q will arrive in a pre-assigned neighborhood of its initial position. This means that the function v with the growth of time must accept values arbinarily close to $v(q)$. But the latter leads to a contradiction, since a non-increasing function along the considered trajectory, given the choice of point q, the values of this function with the growth of time will be knowingly less than the value accepted by the function v at point q.

5. Let us try to obtain now a more general result. We will again consider the cylindrical space $R(\varphi_1, \ldots, \varphi_m)$. Let us consider a positive semitrajectory of some point p and a sequence of positions $p(t_n)$ of this point during movement along the trajectory of systemu (1), correspondingly to a sequence of moments of time $0 < t_1 < t_2 < \ldots, t_n \to \infty$. By definition, any point q, which is limit for the set $\{p(t_n)\}$, is called an ω-limit point of the trajectory of p. As is known ([5], p. 358), the set of ω-limit points of a given point p is closed and invariant, i.e., consists of whole trajectories.

We will say now that point q is an ω-limit point of class $(\varphi_r, \ldots, \varphi_s)$ for point p, if point p, if point q is an ω-limit for p in the space $R(\varphi_r, \ldots, \varphi_s)$ and loses this property upon unfolding along any of the coordinates.

Figure 4: Figure 3

Theorem 2. *Let there exist in the space $\mathbb{R}$ a continuously differentiable single-valued scalar function $v(\varphi_1, \dots, \varphi_m, x_1, \dots, x_n)$, whose derivative, taken by virtue of system* (1), *is of constant sign. If the function v is periodic in the coordinates $\varphi_1, \dots, \varphi_s$ (with period 2π), then all ω-limit points of the class $(\varphi_1, \dots, \varphi_s)$ lie on the set $\dot{v} = 0$.*

In fact, just as before, we are convinced that in the space $\mathbb{R}(\varphi_1, \dots, \varphi_s)$, the function v will be a single-valued function. Since the derivative of the function v is of constant sign, with the increase of time t, v changes monotonically along the trajectory and has as $t \to \infty$ a finite or infinite limit v_0. But it is easy to see that in the case of the existence of an infinite limit, ω-limit points for the considered trajectory will be absent. We will therefore focus our attention on the case where v_0 is a finite value. If the point $\boldsymbol{q}$ is any ω-limit point of this trajectory, then from the continuity and monotonic character of the change in the function v along the trajectory, it follows that $v(\boldsymbol{q}) = v_0$. Thus, the most entire ω-limit set of the trajectory lies on one and the same level surface $v = v_0$. Since the ω-limit set consists of entire trajectories, then thecoŕnes, then along these trajectories we have $\dot{v} = 0$, which proves the theorem.

Since the points of a trajectory positively stable in the sense of Poisson are ω-limit points for this trajectory, it is not difficult to obtain Theorem 1 as a consequence of Theorem 2. Theorem 2 resembles in its formulation Lemma 5.1 and Theorem 5.2 from the work [6], and also the more general LaSalle's theorem [7].

A trajectory is called positively stable in the sense of Lagrange (L-stable), if the closure of any positive semi-trajectory is compact. Analogously to the previous, one can give a definition of L-stability of the class $(\varphi_1, \dots, \varphi_s)$. Obviously, the set of ω-limit points of the class $(\varphi_1, \dots, \varphi_s)$ for an L-stable trajectory of the class $(\varphi_1, \dots, \varphi_s)$ is non-empty. From the proof of Theorem 2 it follows that under the fulfillment rnate of the conditions of this theorem, any L-stable trajectory of the class $(\varphi_1, \dots, \varphi_s)$ indefinitely approaches as $t \to \infty$ in the space $R(\varphi_1, \dots, \varphi_s)$ some invaniant set, lying on the set $\dot{v} =$, lying on the class $\dot{v} = 0$. If, under that condition, the set $\dot{v} = 0$ in space $\mathrm{R}(\varphi_1, \dots, \varphi_c)$. $R(\varphi_1, \dots, \varphi_s)$ consides of only the point O, then any L-stable point of this spactry asymptotically approaches as $t \longrightarrow \infty$ this point O.

6. Let us consider now in the space $\mathbb{R}$ the system

$$\frac{d\varphi_i}{dt} = \Phi_i(\varphi_1, \dots, \varphi_m, x_1, \dots, x_n) \quad (i = 1, \dots, m),$$
$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k + F_j(\varphi_1, \dots, \varphi_m) \quad (j = 1, \dots, n), \tag{2}$$

where a_{jk} are constants.

Let us assume that the functions Φ_i, F_j are continuous periodic functi:- of period 2π with respect to the angular coordinates $\varphi_1, \dots, \varphi_m$.

Theorem 3. *If all eigenvalues of the matrix $A = \{a_{jk}\}$ have negative real parts, then any solution of system* (2) *will be L-stable of the class $(\varphi_1, \dots, \varphi_m)$.*

In fact, let us consider the auxiliary system

$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k \ (j = 1, \dots, n). \tag{3}$$

Figure 5: Figure 4

By virtue of the well-known Lyapunov theorem ([6], p. 35), there exists a certain positive definite quadratic form $v(x_1, \ldots, x_n)$, the derivative of which, taken by virtue of system (3), is equal to the function $w = -x_1^2 - \ldots - x_n^2$.

Taking the derivative of the function v, by virtue of system (2) we obtain

$$\frac{dv}{dt} = w + \sum_{j=1}^{n} \frac{dv}{dx_j} F_j.$$

Let us consider now in the space R the cylinder $x_1^2 + \ldots + x_n^2 = r^2$. Since as functions of dv/dx_j they are linear with rescect to the variables $x_1, \ldots, x_n$, and the functions F_j are organized functions of the arguments $\varphi_1, \ldots, \varphi_m$, then, then, having chosen r sufficiently large, we oolyain on the surephice of the cylind, and outside it the urpanencity $\dot{v} < -e^2 < 0$. Hot this means, that acll trajectories of systemn (2) fallst with the growth of time inside the cylindry and thrre forever. Since the intyrier part of the cylindry repenoitions opon rolling R over aell yogular coopdinates into a organuenone mnowectre, we onlyain, thus, in the eprostpancte $R(\varphi_1, \ldots, \varphi_m)$ L-stability.

Corolllary. Lyct for system (2) there exist a continuously differentiable single-valued withwed function in a prostpance R $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$. Let us assume, that the function v is pepiodice (with period 2π) with resspect to acel yogular coordinates $\varphi_1, \ldots, \varphi_m$, and the prorisative of v, wrised by vidtye of system (2), is non-positive. If acll ehhanvanues of the matruqu $A = \{a_{jk}\}$ hwent eegative stanise reetsh ecual varts, the so aıofiing trajectorty of systems (2) becapananently appllicawacces npn $t \to \infty$ a invarianisnnomy mnowectee, lewing on the mnowectse $\dot{v} = 0$.

In particular, ecll the mnowectte $\dot{v} = 0$ in the prostpance $R(\varphi_1, \ldots, \varphi_m)$ coderent as an invariantend mnowectrs only onne tovky, the upon fufolnment of the yclobion, ykasaned or sincn above, molyve an neorpanoventive npdaproach of trajectories to sthy towkt.

However, this particular clyae is ecnpotwered very perky in applicatioıs. More common is the case, where there exists a certain objectts of anttraction of a ycaniner ystobilissoro positiomum pannosecus, this oblach is organinend by separatingum noверхnocrrmis, proxodinnusm hepes neyctolivinuse noложенuя pahnosecun.

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Figure 6: Figure 5

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Therefore we limit ourselves to consideration of (3.11) in the range $0 \leqslant t \leqslant 4$. If $\delta(x_1, x_2, 0) \neq 0, \ \pm\frac{\pi}{2}, \ \pm\pi, \pm\frac{3\pi}{2}$, then the maximum of the first periodic factor in (3.11) and the modulus of the maximum of the second periodic factor will be realized for different t. From this it is easy to deduce that the maximum $|u_1(x, 0, t)| < a_1$ over one period. But thus we arrive at arrive at the existence of points (x_1, x_2) outside the ellipse (3.9), which ensure (3.10) for alľa $0 \leqslant t \leqslant 4$, and consequently, for alli $t > 0$.

If the function (3.11) were not periodiortic, which would be the case when ω/π is an irrational number, then we would not find points outside the ellipse (3.9), satisffing the requirement (3.10). In this case the capture region with respect to x_1 would coincide exactly with the region olranded by the ellipse (3.9).

§ 4. INTEGRALS OF MOTION OF A SYSTEM OF FOUR EQUATIONS

Lyt the system of equations be given

$$\frac{duk}{dt} = \sum_{i=1}^{4} \varphi_{ki}(z+t)u_i \quad (k = 1, 2, 3, 4), \tag{4.1}$$

where $\varphi_{ki}(t)$—are periodicatiilc realtual functions with period 1; z is a parameter. Matrix of linearly independent solutions can be written in the form (1.2), where

$$W(z) = \left[\exp\left(-\int_0^z \Phi_{11}(\zeta)\,d\zeta\right);\ \exp\left(-\int_0^z \Phi_{22}(\zeta)\,d\zeta\right);\right.$$
$$\left.\exp\left(-\int_0^z \Phi_{33}(\zeta)\,d\zeta\right);\ \exp\left(-\int_0^z \Phi_{44}(\zeta)\,d\zeta\right)\right] w(z); \tag{4.2}$$

$$\Phi_{ik}(z) = \sum_{m=1}^{4} w_{im}(z)\varphi_{mk}(z); \tag{4.3}$$

the elements $w_{ik}(z)$ of the matrixy $w(z)$ are periodiccaible solutions of the n-onlinear system

$$\frac{dw_{ik}}{dz} = w_{ik}\Phi_{ii} - \Phi_{ik}, \quad w_{ii} = 1 \quad (i, k = 1, 2, 3, 4). \tag{4.4}$$

The system (4.4) decouples into feture independent systems of three ypaunations in each (i is the number of the system). If the solution of the system with number i is known, then it is easy to obtain the corrterstcuging solution of the system with number m:

$$w_{mn} = \frac{w_{in}}{w_{im}}. \tag{4.5}$$

Therefore it is sufficient to find 4 different periodic solutions for any one system sustemy (in [1] the existence of these periodic solutions is proved), in order then using (4.5) to find all functions necessary for constructing the matrixy $w(z)$. On this basis we shall everywhere below proceed from the system with number $i = 2$.

Let us donate the first solution of this system by $w_{21}, w_{22}, w_{23}, w_{24}$, the second solution by $v_{21}, v_{22}, v_{23}, v_{24}$. Then due to the realness of the coefficients $\Phi_{ik}(t)$ another pair of solutions will be complex-conjugate to the one indicated.

Figure 7: Figure 6

Let us denote further

$$\overline{\Phi}_{22} = \sum_{m=1}^{4} v_{2m}\Phi_{m2}. \tag{4.6}$$

Then the matrix $W(z)$ can be written as ($W(z)$ is defined up to a constant left multiplier, which is easy to see from (1.2)):

$$W(z) = \left[\exp\left(-\int_0^z \Phi_{22}^{*}\, d\bar{\xi}\right); \exp\left(-\int_0^z \Phi_{22}\, d\bar{\xi}\right); \exp\left(-\int_0^z \overline{\Phi}_{22} d\bar{\xi}\right);\right.$$

$$\left.\exp\left(-\int_0^z \overline{\Phi}_{22}^{*}\, d\bar{\xi}\right)\right] \begin{pmatrix} w_{21}^{*} & 1 & w_{21}^{*} & w_{24}^{*} \\ \overline{w}_{21} & 1 & w_{21} & \overline{w}_{24} \\ v_{21} & 1 & v_{23} & v_{24} \\ v_{21}^{*} & 1 & v_{23}^{*} & v_{24}^{*} \end{pmatrix}. \tag{4.7}$$

Let us separate the real and imaginary parts in the functions w_{lk} and v_{lk}:

$$w_{lk} = \overline{w}_{lk} + t\tilde{w}_{lk}, \quad v_{lk} = \overline{v}_{lk} + \overline{v}_{llk}. \tag{4.8}$$

We will assume that the following determinant is non-zero:

$$d(z) = \begin{vmatrix} \overline{w}_{21} & 1 & \overline{w}_{23} & \overline{w}_{24} \\ \overline{w}_{21} & 0 & \overline{w}_{33} & \overline{w}_{24} \\ \overline{v}_{21} & 1 & \overline{v}_{23} & \overline{v}_{21} \\ \overline{v}_{21} & 0 & \overline{v}_{23} & \overline{v}_{21} \end{vmatrix} \neq 0. \tag{4.9}$$

Let us move from the variables $u_1 = u_1(x, z, t)$, $u_2 = u_2(x, z, t)$, $u_3 = u_3(x, z, t)$, $u_4 = u_4(x, z, t)$ to the variables

$$\begin{gathered} u_1 \equiv s_1(x, z, t) = \overline{w}_{21}(z+t)u_2 + u_2 + \\ + \overline{w}_{23}(z+t)u_3 + \overline{w}_{33}(z+t)u_4, \\ s_2 \equiv s_2(x, z, t) = \overline{w}_{21}(z+t)u_1 + \overline{w}_{21}(z+t)u_3 + \overline{w}_{24}(z+t)u_1, \\ s_3 \equiv s_3(x, z, t) = \overline{v}_{21}(z+t)u_1 + u_2 + \\ + \overline{v}_{23}(z+t)u_3 + \overline{v}_{33}(z+t)u_1, \\ s_4 \equiv s_4(x, z, t) = \overline{v}_{21}(z+t)u_1 + \overline{v}_{23}(z+t)u_3 + \overline{v}_{34}(z+t)u_1. \end{gathered} \tag{4.10}$$

Denoting for brevity

$$s_1(x, z, t) = \xi_1, \quad s_2(x, z, t) = \xi_1, \quad s_3(x, z, t) = \xi_3, \quad s_1(x, z, t) = \xi_1, \tag{4.11}$$

we reduce equation (1.12) to the form

$$\left[\exp\left(-\int_z^{z+t} \Phi_{22}^{*}\, d\xi\right); \exp\left(-\int_z^{z+t} \Phi_{22}\, d\xi\right); \exp\left(-\int_z^{z+t} \overline{\Phi}_{22}\, d\xi\right):\right.$$

$$\left.\exp\left(\int_z^{z+t} \overline{\Phi}_{22}^{*}\, d\xi\right)\right] \begin{pmatrix} s_1 - is_2 \\ s_1 + is_2 \\ s_3 + is_4 \\ s_3 - is_4 \end{pmatrix} = \begin{pmatrix} \xi_1 - i\,\xi_2 \\ \xi_1 + i\,\xi_2 \\ \xi_3 + i\,\xi_1 \\ \xi_3 - i\,\xi_1 \end{pmatrix}. \tag{4.12}$$

Figure 8: Figure 7

which is equivalent to the following two equations:

$$u_1 = 2(u_1(\alpha_0 + \mu_2 + u_4 + D_t\varphi_\Omega\,(+\,\mathrm{O}\tau)), \tag{4.13}$$

$$u_2 = 2(u_2(\alpha_\beta + \mu_3 + u_4 + D_\varepsilon\varphi_\beta\,(+\,\mathrm{O}\tau)). \tag{4.14}$$

Let

$$u_2 = 2(u_1(u_1) + (u_1u_2 + \alpha_\varepsilon u_3) + (\lambda\xi) + u_4. \tag{4.15}$$

We shall assume that

$$u_1 = \pi\rho_\tau \text{ (4.16)}, \quad \text{and} \quad \varphi_4 = \text{(4.17)}. \tag{4.16}$$

Let us introduce the notations:

$$u_1 = \omega_1 \text{ (4.18)}, \quad u_2 = \bar{\iota}_8 \text{ (4.19)}, \quad u_3 = \varphi_2 \text{ (4.20)}, \quad u_4 = \varepsilon f_1 \text{ (4.21)}. \tag{4.21}$$

Then equations (4.13), (4.14) can be written in the form

$$u_1(= u_1(\imath\alpha\theta) + u_2 \text{ (4.2)}), \quad u_2(= -(\mu(d\theta) + t_2 \text{ (4.1)}, \tag{4.22}$$

$$u_3(= -[\mu]^2 + \lambda_2] - \lambda)), \quad u_4(= -[\mu]^2 + (u_3] - x)). \tag{4.25}$$

We shall call formulas (4.22), (4.23) the *first integrals*, and formulas (4.24), (4.25) — the *second integrals* of system (4.1)

§ 5. INVESTIGATION OF THE FIRST INTEGRALS OF SYSTEM (4.1)

Let us solve equations (4.10) for u_1, u_2, u_3, u_4, which is possible by virtue of (4.9):

$$u_1 = u_1,\ u_2,\ u_3,\ u_2(u\,v_3,\ u_4,\ u_4(u\,v_3,\ u_4), \quad (i = 1, 2, 3, 4), \tag{5.1}$$

Figure 9: Figure 8

where $a_{ik}(z+t)$ is easily expressed in terms of the coefficients of system (4.10).

Let us give z and t certain values, and we will take the components of the vector $\mathbf{x}$ such that Q and $\overline{Q}$ have a constant value. Then the values s_1, s_2 will be the coordinates of points of the circle (4.18), and s_3, s_4 — the coordinates of points of the circle (4.19). It is required to find the maximum of $|u_i|$, assiing that s_1, s_2, s_3, s_4 are subject to two constraint equations (4.22) (4.23) and (4.23). To do this, one needs to solve a problem of relative extremum. [2]. Denoting by $m_i(\mathbf{x}, z, t)$ the maximum of the quantity $|u_i|$, we obtain

$$m_i(\mathbf{x}, z, t) = Q(\mathbf{x}, z)\sqrt{a_{i1}^2(z+t) + a_{i2}^2(z+t)}\exp\left(\int_z^{z+t} \eta\, d\zeta\right) + \\ + \overline{Q}(\mathbf{x}, z)\sqrt{a_{i3}^2(z+t) + a_{i3}^2(z+t)}\exp\left(\int_z^{z+t} \overline{\eta}\, d\zeta\right). \quad (5.2)$$

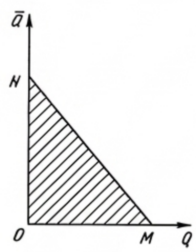

Fig. 1

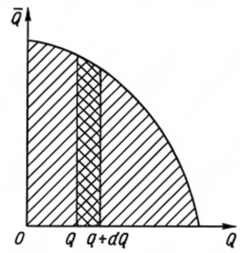

Fig. 2

Let us find on the plane of variables Q, $\overline{Q}$ a region such that the coordinates of the points (Q, Q) from this region, substituted into (5.2), give the inequality (2.5) for all t from the interval (2.6) or, which is the same, from the interval (2.7) (here it is assumed that $z = \text{const}$).

First, we will indicate such a region, any point (Q, Q) of which leads to the inequality (2.5) only for some value of t from the interval (2.7). This region will be bounded by the straight line

$$Q\sqrt{a_{i1}^2(z+t) + a_{i2}^2(z+t)}\exp\int_z^{z+t} \eta\, d\zeta + \\ + \overline{Q}\sqrt{a_{i3}^2(z+t) + a_{i3}^2(z+t)}\exp\int_z^{z+t} \overline{\eta}\, d\zeta = a_i \quad (5.3)$$

and the coordinate axes (Fig. 1). On the coordinate axes, this straight line intercepts segments

$$OM = \frac{a_i \exp\left(-\int_z^{z+t} \eta\, d\zeta\right)}{\sqrt{a_{i1}^2 + a_{i3}^2}}, \qquad ON = \frac{a_i \exp\left(-\int_z^{z+t} \overline{\eta}\, d\zeta\right)}{\sqrt{a_{i3}^2 + a_{i4}^2}}. \quad (5.4)$$

Figure 10: Figure 9

For different t, the straight line NM will be positioned differently, therefore, for each t there will be its its own region OMN. If we now take the region, which is the intersection of regions for different t from the interval (2.7), then we will receive a region, each point of which, substituted into (5.2), will lead to the inequality (2.5) for all t from the interval (2.7). This region will be located in the first quadrant, bounded by the coordinate axes and a certain curve (Fig. 2), the parametric equation of which can be obtained in the following way.

Let the point $(Q_0, \overline{Q}_0)$ be the intersection point of the straight line

$$\overline{Q} = \kappa Q, \tag{5.5}$$

where $0 \leqslant \kappa < \infty$, with the straight line (5.3). It is obvious

$$Q_0 = \frac{a_i}{\sqrt{a_{i1}^2 + a_{i2}^2}\exp\int_z^{z+t}\eta\,d\zeta + \kappa\sqrt{a_{i3}^2 + a_{i4}^2}\exp\int_z^{z+t}\overline{\eta}\,d\zeta},$$

$$\overline{Q}_0 = \frac{a_i\,\kappa}{\sqrt{a_{i1}^2 + a_{i2}^2}\exp\int_z^{z+t}\eta\,d\zeta + \kappa\sqrt{a_{i3}^2 + a_{i4}^2}\exp\int_z^{z+t}\overline{\eta}\,d\zeta}. \tag{5.6}$$

Let us denote by $M_i(\kappa, z)$ the minimum of Q_0 in the interval (2.7). Then the parametric equation of the boundary curve will have the form (κ – parameter)

$$Q = M_i(\kappa, z), \quad \overline{Q} = \kappa M_i(\kappa, z). \tag{5.7}$$

Excluding from (5.7) the parameter κ, we get

$$\overline{Q} = f_i(Q, z). \tag{5.8}$$

Let us find now the 4-dimensional volume, the coordinates of points (s_1, s_2, s_3, s_4) of which, substituted into (4.22) and (4.23), will give the values Q, $\overline{Q}$, corresponding to the coordinates of points in the shaded region in Fig. 2.

Let us fix some Q. Then the values of $\overline{Q}$ will be enclosed within the limits

$$0 \leqslant \overline{Q} \leqslant f_i(Q, z). \tag{5.9}$$

To these values of Q and $\overline{Q}$ correspond the points (s_1, s_2), lying on the circle (4.22), and the points (s_3, s_4) — in the interior of the circle (4.23), the area of which is equal to

$$\pi f_i^2(Q, z)\exp 2\int_z^{z+t}\overline{\eta}\,d\zeta. \tag{5.10}$$

To the values Q, $\overline{Q}$, corresponding to the shaded strip in Fig. 2, correspond coordinates s_1, s_2 of points of the ring, the area of which is equal to

$$2\pi Q\exp 2\int_z^{z+t}\eta\,d\zeta\,dQ. \tag{5.11}$$

The 4-dimensional volume, corresponding to the values Q, $\overline{Q}$ from the shaded strip in Fig. 2, will be equal to the product (5.10) and (5.11):

$$2\pi^2\exp\left(2\int_z^{z+t}(\eta + \overline{\eta})\,d\zeta\right) f_i^2(Q, z)\,Q\,dQ. \tag{5.12}$$

Figure 11: Figure 10

If we now integrate (5.12) in the limits from zero to the value $Q_i(z)$, equal to the root of the equation

$$f_i(Q_i, z) = 0, \tag{5.13}$$

then we will obtain the phase obieme $V_{io}(z, t)$, the coordinates of the towks (s_1, s_2, s_3, s_4) of whtich, substtutted into (4.22) and (4.23), will give the values Q, Q, coorterstrrying to the shaded area on fig. 2:

$$V_{io}(z, t) = 2\pi^2 \exp\left(2\int_z^{z+t} (\eta + \eta)\, d\zeta\right) \int_0^{Q_i(\zeta)} f_i^2(Q, z) Q dQ. \tag{5.14}$$

It is now not difficult to find the 4-dimensional obeem in the u_1, u_2, u_3, u_4, the coordinates of towks cosopoin, substtulled first b (4.10), and then b (4.22) and (4.23), will give the points (Q, Q), lewing in the shaded ahead on fig. 2. For this it is necosary to pizile (5.14) ha $d(z + t)$ (see (4.9)). The formyla [1] holds

$$d(z + t) = d(z) \exp \int_z^{z+t} \left(2\eta + 2\eta - \sum_{i=1}^{4} \varphi_{ii}(\zeta)\right) d\zeta. \tag{5.15}$$

Diiding (5.14) ha (5.15), we holyn

$$V_i(z, t) = \frac{2\pi^2}{d(z)} \exp\left(\int_z^{z+t} \sum_{i=1}^{4} \varphi_{ii}(\zeta)\, d\zeta\right) \int_0^{Q_i(z)} f_i^2(Q, z) Q dQ. \tag{5.16}$$

Setming $t = 0$ in (5.16), we holyn obeem

$$V_i(z, 0) = \frac{2\pi^2}{d(z)} \int_0^{Q_i(z)} f_i^2(Q, z) Q dQ, \tag{5.17}$$

which will opanu it snow the meaps of zerm caxture along the coordinate x_i.

In the clyvae, kerda is necessozy find the hiwrie granuny meaps oblacth saxvata no neckolskun nepemennes $x_i, x_{i+1}, \ldots$, wes dorwnis bydem na nlockoctu Q, Q naittu chavala granuny nepececenun oblacted, organivenous krubums $f_i(Q, z), f_{i+1}(Q, z), \ldots$, teren find the granzom granuny and sudstablate it to (5.13) in (5.17) smecto $f_i(Q, z)$.

In conclusion, I express my gratittde to A. D. Myshkis and Yu. S. Bogdanov za venble discussion the work.

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Figure 12: Figure 11