INVARIANT COVERINGS OF SUBGROUPS

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INVARIANT COVERINGS OF SUBGROUPS

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In paper (1) P. G. Kontorovich introduced into consideration groups every cyclic subgroup of which is contained in a proper invariant subgroup. They were called invariantly coverable groups. It was also shown there that a finite group is invariantly coverable if and only if the factor group by the commutator subgroup is a noncyclic group. In the present paper we investigate finite nonnilpotent groups every proper subgroup with a certain property of which is contained in a proper invariant subgroup.

We shall need the following definitions.

A group (subgroup) is called a t-group (subgroup) if the property t holds in it. Otherwise the group (subgroup) is called a $\bar{t}$-group (subgroup). We shall deal only with absolute properties t, i.e., those preserved in a group (subgroup) under any monomorphism of it into another group. We shall call a property t a special property if every t-group is nilpotent, every subgroup of a t-group (the t-group itself and every homomorphic image of a t-group) is a t-group. For example: the properties "to be an abelian group," "to be a nilpotent group," "to be a nilpotent group with nilpotency class not exceeding k," "to be a primary group." A group is called an (i,t)-group if in it every proper t-subgroup is contained in a proper invariant subgroup. A group is called an $(i,\bar{t})$-group if in it every proper $\bar{t}$-subgroup is contained in a proper invariant subgroup. Otherwise the group is called, respectively, an $(\bar{i},t)$-group, an $(\bar{i},\bar{t})$-group. A nilpotent group is always an (i,t)-group and an $(i,\bar{t})$-group. Throughout this paper $(i,\bar{t})$-groups and $(\bar{i},\bar{t})$-groups are always nonnilpotent, and t is a special property.

1. *A homomorphic image of an $(i,\bar{t})$-group is an $(i,\bar{t})$-group.*
2. *In an $(i,\bar{t})$-group the normalizer of a noninvariant Sylow subgroup is a maximal t-subgroup, maximal in the group.*
3. *An $(i,\bar{t})$-group is solvable.*

We shall prove this proposition using induction on the order of the group. Suppose that the group has no nonidentity invariant primary subgroups. Then, applying 2 and the known theorem of J. Thompson on invariant p-complements

([2]), we show that the group is nilpotent. Hence the group has an invariant, nonidentity primary subgroup. The factor group by it is an $(i, \bar{t})$-group by 1 and, by the induction hypothesis, is solvable. But then the group itself is solvable.

4. *In a solvable $(i, \bar{t})$-group G one has $Z(G) = E$ if and only if $G = K\lambda H$, where $K \lhd G$ (K is a minimal invariant subgroup), K is a maximal nilpotent subgroup, $H \neq E$ is a maximal t-subgroup.*

Proof. Necessity. In G there is a $K \lhd G$, $o(K) = p^{\alpha}$. $K \leq P$, where P is a Sylow p-subgroup.

It is known that the intersection of a nonidentity invariant subgroup of a nilpotent group with its center is nonidentity.

The group G contains at least one Sylow subgroup Q such that $K \not\subseteq N(Q)$. Applying 2, we obtain that $G = KN(Q)$; moreover, $G = K\lambda N(Q)$, and if $H = N(Q)$, then $H \neq E$ is a maximal t-subgroup, maximal in the group G.

Next, since $K \lhd G$, we have $C(K) \lhd G$ and $C(K) \supseteq K$. Suppose that K does not coincide with $C(K)$. Then $C(K) \cap H = T \neq E$ and $T \lhd H$. Since t is a special property, $T \cap Z(H) = L \neq E$. This means that $L \subseteq C(H)$ and $C(K)$, i.e. $L \subseteq Z(G)$.

Thus, $K = C(K)$. Since H is nilpotent, $P \lhd G$. Hence $Z(P) \lhd G$. Since $K \lhd G$ and $K \cap Z(P) \neq E$, it follows that $K \subseteq Z(P)$, and therefore K is a Sylow p-subgroup; moreover, K is a maximal nilpotent subgroup in G.

The sufficiency is almost obvious.

5. Theorem. Let $\Gamma(G)$ be the hypercenter of the group G. A nonnilpotent group G is an $(i, \bar{t})$-group if and only if

$$G/\Gamma(G) \cong B = K\lambda H,$$

where $K \lhd B$, K is a maximal nilpotent subgroup in B, $H \neq E$ is a maximal t-subgroup in B, and the complete inverse image of H is a t-subgroup in G.

We outline the proof of the theorem.

The necessity is given by Proposition 4.

Sufficiency. Let D be a maximal $\bar{t}$-subgroup in G. Since t is a special property, D is a maximal subgroup in G.

If $D \not\supseteq \Gamma(G)$, then, since $G = \{\Gamma(G), D\}$, it is easy to show that $D \lhd G$.

Let $D \supset \Gamma(G)$, and let φ be a homomorphism of G onto B. Suppose that $\varphi(D) \not\supseteq K$. Then $B = K\lambda\varphi(D)$. Since $\varphi(D)$ is conjugate to H in B, D, as the complete inverse image of $\varphi(D)$, is conjugate to the complete inverse image of H, and therefore is a t-subgroup in G, which contradicts the choice of D. Thus, $\varphi(D) \supseteq K$, $\varphi(D) = K\lambda[\varphi(D) \cap H]$. Since t is a special property and $\varphi(D)$ is a maximal subgroup in B, we have $\varphi(D) \lhd B$, and therefore $D \lhd G$.

6. Corollary. If the group G is an $(i, \bar{t})$-group, then

$$G = P\lambda[Q \times \dots \times S],$$

where $P, Q, \dots, S$ are Sylow subgroups of G.

7. Corollary. In a nonnilpotent group G, every nonnilpotent maximal subgroup is invariant if and only if

$$G/\Gamma(G) \cong B = K\lambda H,$$

where $K \lhd B$, K and $H \neq E$ are maximal nilpotent subgroups in B.

We note that the class of groups indicated in 7 is not incident with the class of polynilpotent groups introduced in (3). The intersection of these classes includes the class of groups all of whose subgroups are nilpotent (4).

8. Corollary. In a group G, every nonprimary maximal subgroup is invariant if and only if

$$G = P\lambda Q,$$

where $P \lhd G$; P and Q are Sylow subgroups in G.

9. Corollary. Let every nonnilpotent subgroup in a nonnilpotent group G be invariant. Then

$$G/\Gamma(G) \cong B = [\{a_1\} \times \dots \times \{a_d\}]\lambda\{b\},$$

where B is a Frobenius group,

$$a_1^p = \dots = a_d^p = b^n = 1; \qquad n = q_1^{\beta_1} \cdots q_l^{\beta_l}; \qquad p^d \equiv 1 \pmod{n};$$

d is the least exponent for which

$$p^d \equiv 1 \pmod{q_i}$$

for each $i = 1, \dots, l$; $p, q_1, \dots, q_l$ are prime numbers.

The proof is obtained with the help of 7 and known results on Frobenius groups.

We indicate a sufficient criterion for $(\bar{i}, \bar{t})$-groups.

10. Theorem. Let $\mathfrak{A}$ be a class of groups G such that every nilpotent subgroup of G is a t-subgroup. Then every nonnilpotent group in $\mathfrak{A}$ is an $(\bar{i}, \bar{t})$-group.

Proof. Suppose there exists a non-nilpotent (i, t)-group in the class $\mathfrak{A}$. Among such groups choose a group G of least order. G contains a maximal subgroup M that is not invariant in G. From the condition of the theorem it follows that M is a t-subgroup. It can be shown that M is a non-nilpotent (i, t)-group. Obviously, $M \in \mathfrak{A}$, but the order of M is less than the order of G, which contradicts the choice of G from $\mathfrak{A}$.

11. Let the property t be such that every nilpotent group is a t-group, and let $\mathfrak{A}$ be the class of all groups.

Corollary. A group is nilpotent if and only if it is an (i, t)-group. In particular, a group is nilpotent if and only if every proper nilpotent subgroup is contained in a proper invariant subgroup.

12. Corollary. Every non-nilpotent group is the composite of some nilpotent subgroup and its conjugates.

13. Let $\mathfrak{A}$ be a class of groups all of whose Sylow subgroups are abelian.

Corollary. Every non-abelian group from $\mathfrak{A}$ contains an abelian subgroup that is not contained in any proper invariant subgroup, and is the composite of this abelian subgroup and its conjugates.

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Note: Figure translations are in progress. See original paper for figures.