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## Abstract

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**PHYSICS**

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# STOCHASTIC DESTRUCTION OF MAGNETIC SURFACES OF A STELLARATOR

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§ 1. The main feature of stellarator fields is the presence of an average angle of rotation of the lines of force, $\omega(r)$, which usually depends on the (small) radius. Therefore the "behavior" of the lines of force in a stellarator is analogous to the motion of a nonlinear oscillator. Owing to the closedness of the stellarator, perturbations have a period $L$ (the perimeter of the stellarator). The most dangerous perturbations are resonant ones. From the numerous works on the investigation of such perturbations (see, for example, ([1–3])) one may get the impression that increasing the nonlinearity ($d\omega/dr$) always leads to enhanced stability. Similar hopes also existed at the initial stage of development of strong-focusing accelerators. Although nonlinearity does stabilize resonances, it simultaneously leads to the appearance of new instabilities. The most dangerous of them is apparently the so-called stochastic instability ([4–5]). When it arises, the lines of force are arranged in a quasi-random manner and relatively quickly leave the separatrix.

As applied to a stellarator, stochastic processes were first considered by Sagdeev and Zaslavskii. In the present work the main attention is devoted to the criterion of stochasticity, and also to the use of this instability for creating Skorniakov' s trap ([2]).

§ 2. We shall start from the Hamiltonian equations of "motion" of magnetic lines of force ([6]):

$$ds/dz = 2\varepsilon n s^{n/2} \sin n\theta, \qquad \theta = \varphi - \alpha z, \qquad s = (r/a)^2,$$

$$d\varphi/dz = \varepsilon n s^{n/2-1} \cos n\theta, \qquad \varepsilon a = 4I/caH_z. \tag{1}$$

Here $n$ is the number of starts of the helical field with pitch $2\pi/\alpha$; $a$ is the radius of the cylinder on which $2n$ conductors with current $I$ in each are located; $H_z$ is the longitudinal field; $r, \varphi, z$ are cylindrical coordinates. The equations are valid for $\varepsilon a$, $s \ll 1$. However, the order-of-magnitude estimates that constitute the main content of the work are valid in a wider region, in fact everywhere except the immediate vicinity of the separatrix. The same remark also applies to the other strong inequalities in this work.

Equations (1) describe both the principal field of the stellarator ($\varepsilon = \text{const}$) and the (time-independent) perturbations (with parameters $\varepsilon_1, n_1, \alpha_1$). Suppose that there is a set of short ($a \ll l \ll (n_1\alpha_1)^{-1}$) uncorrelated perturbations, i.e., the perturbation parameters $\varepsilon_1, n_1, \alpha_1$ are constant over a length $l$ (the correlation length) and are independent in neighboring sections.

Let us first consider the action of one such section. Owing to the periodicity of the perturbation, the system can be described by difference equations

$$s_{N+1} = s_N \left(1 + \frac{2}{n_1}\xi_N \sin\psi_N\right),$$

$$\psi_{N+1} = \psi_N + \alpha L_1 \omega(s_{N+1}) + \xi_N \cos\psi_N, \tag{2}$$

$$L_1 = n_1 L, \qquad \psi_N = n_1\varphi_N, \qquad \xi_N = \varepsilon_1 n_1^2 l s_N^{n_1/2-1},$$

where $N$ is the number of turns around the stellarator; $s_N, \psi_N$ are the coordinates of the field line in the perturbation zone, and the magnetic surface is assumed to be cylindrical ($s = \text{const}$). Equations (2) are obtained by direct integration of (1) for a short perturbation; the term $aL_1\omega$ describes the mean twisting of the field line in the unperturbed region.

§ 3. The behavior of the solution of (2) depends qualitatively on the phase-stretching parameter ($\xi \ll 1$):

$$K = d(\psi_{N+1} - \psi_N)/d\psi_N = \Omega_\phi^2 \cos\psi_N,$$
$$\Omega_\phi^2 = 2aLs\omega'\xi, \qquad \omega' \equiv d\omega(s)/ds.$$
$$\omega' \equiv d\omega(s)/ds. \tag{3}$$

For $|K| \ll 1$, the difference equations (2) may be replaced by the differential equations

$$\dot{s} = -\frac{2}{n_1} s\xi \sin\psi, \qquad \dot{\psi} = aL_1(\omega - \omega_p) + \xi\cos\psi, \tag{4}$$

where $\omega_p = 2\pi m/aL_1$ ($m = 0, 1, 2, ...$) are the resonant values of $\omega$ (3)*; the dot denotes differentiation with respect to $N$.

Equations (4) describe a nonlinear resonance. For sufficiently small $\xi$, $s$ performs oscillations with amplitude[8]

$$\Delta s/s \sim \sqrt{\xi/aLs\omega'}. \tag{5}$$

To stabilize the peripheral resonance ($\omega_p \neq 0$), it is necessary that the nonlinear detuning (4) $aL_1(\omega - \omega_p) \sim \Omega_\phi$ (8) exceed the resonance width $\xi$:

$$\xi \lesssim 2aLs\omega' = 2(n-2)aL\omega = 4\pi(n-2)N_l(L), \tag{6}$$

where $N_l(L)$ is the number of turns of the field line in going around the stellarator (the rotational transform).

The stabilization condition for the central resonance ($\omega_p = 0$) is obtained from (4) in an analogous way and has the form**

$$\xi \lesssim aL_1\omega = 2\pi n_1 N_l(L). \tag{7}$$

Estimate (7) agrees with the results of studies of analogous resonances in accelerator theory ([7], [14]).

§ 4. In the opposite limiting case $\Omega_\phi^2 \gg 1$, the transition from the difference equations (2) to the differential equations (4) is impossible, and transformation (2) is now (with respect to the phase) a stretching almost everywhere, with the exception of narrow regions near $\cos\psi = 0$ ($\Delta\psi \sim \Omega_\phi^{-2}$). If these regions are neglected, which from the physical point of view seems quite justified, then one may apply the results of work ([9]), according to which a dynamical system with stretching is a so-called $K$-system ([10]); the latter possesses all the currently known attributes of stochasticity: ergodicity, mixing, and positive Kolmogorov entropy.

The question may also be approached somewhat differently. It is easy to show that the solution of (2) is locally unstable at every point under the condition:

$$|K| > 4. \tag{8}$$

* Higher-order resonances

$$\omega^{(p)} = \left[(2\pi m/aL) \pm pn\right]/(n_1 \pm pn), \qquad p = 1, 2, \ldots \; ([1]),$$

contain an additional small factor of the form

$$\left(\frac{\xi}{\alpha} s^{n/2-1}\right)^p$$

and can be significant only near the separatrix.

** Note that the central resonance is the most dangerous one, since it leads to destruction of a region of size $r \sim \sqrt{\xi_1^{1/(2n-3)}}$ (for $n_1 = 1$), whereas for a peripheral resonance $\Delta r \sim \sqrt{\xi_1}$ (5). This is an argument in favor of choosing a double-entry helical field, for which $\omega(0) \neq 0$ and can be chosen between resonances.

The term "locally unstable" means, in the present case, that the characteristic numbers of the matrix of the linearized transformation (2) are $|\lambda_i| \neq 1$ ($\lambda_1\lambda_2 = 1$). For $\Omega_\phi^2 \gg 1$ this condition is satisfied everywhere except for the same narrow regions with $\cos\psi \approx 0$. If these are again neglected, then from the results of Refs. ([11,12]) we arrive at the previous conclusion.

Finally, one more approach, connected with the calculation of time correlations, was used in ([13]) for a system similar to (2). In ([13]) the results of numerical integration of difference equations are also given, from which it follows that the boundary of stochasticity corresponds to $|K| = 4$, in agreement with (8).

On the basis of what has been said, we define the boundary of stochasticity by the estimate $\Omega_\phi^2 \sim 4$, or

$$\xi \sim 2/aLs\omega' = 1/\pi(n-2)N_\pi(L). \tag{9}$$

Comparing (6) and (9), we see that the admissible perturbation passes through a maximum in the region

$$N_\pi(L) \sim 1/2\pi(n-2), \qquad \xi_{\max} \sim 1. \tag{10}$$

In the stochastic region the law of "motion" of the field lines has the usual diffusive character and is described by a certain kinetic equation. A simple estimate of the diffusion can be obtained directly from (2)

$$\overline{(\Delta s)^2} \sim \frac{\xi^2}{n_1^2}\, s^2 N. \tag{11}$$

§ 5. Returning to the more general case of a continuous perturbation, with a short correlation length $l$ (§ 2), we can make the usual root-mean-square estimate ([14])

$$\xi \sim \varepsilon_1 n_1^2 s^{n_1/2-1} l\sqrt{L}. \tag{12}$$

As an example, let us give the parameters of the perturbation due to the displacement of the conductors of the helical field by an amount $\delta$: $n_1 = 1, \quad \alpha_1 = \alpha, \quad \varepsilon_1 \sim \dfrac{\varepsilon\delta}{2a}\sqrt{2n}$.

To avoid misunderstandings, we emphasize that the lack of correlation of the perturbations on neighboring segments has no relation to stochastic instability, since the perturbation is exactly repeated after the period $L$.

§ 6. Stochastic instability of field lines can be used to create a Skornyakov trap ([2]), inside which there is a "turbulent" (stochastic) region surrounded by a "laminar" region of regular magnetic surfaces. For ordinary perturbations (§ 2), the quantity $\Omega_\phi^2$ grows rapidly with radius, so that the stochastic region extends to the separatrix, which leads to a violation of thermal insulation. This difficulty can be circumvented by introducing a special resonant winding of length $l(\ll L)$, the helical pitch of which coincides with the pitch of the field line at some radius: $\alpha_1 = \alpha\omega(s_1)$. As a result, the perturbation will act effectively only in a certain band $\Delta s$:

$$\Omega = n_1 a l \omega' \Delta s = 2\pi n_1 (n-2) N_\pi(l) \Delta s/s \sim 1. \tag{13}$$

All the previous relations will remain unchanged with the new $\xi$:

$$\xi = \varepsilon_1 n_1^2 s^{n_1/2-1} l f(\Omega), \tag{14}$$

where $f(\Omega)$ is a resonance factor which, for a certain design of the winding, may decrease sufficiently rapidly with $\Omega$, for example, $f(\Omega) = \exp\{-|\Omega|\}$. This allows one to hope that the external "laminar" layer will be sufficiently reliable.

An analysis of the optimal parameters of the resonant winding shows that $\Omega_\phi^2 \lesssim 2L/l$; together with estimate (13), this leads to the necessity of having a relatively long stellarator, at least $\pi(n-2)N_\pi(L) \gtrsim 1$.

Adjustment of the position and size of the stochastic region can be carried out by changing the currents in the main and resonant windings of the stellarator.

§ 7. The point of Skornyakov' s trap lies in the hope that the "turbulent" (stochastic) region of the magnetic field will help suppress plasma instabilities. An investigation of this problem lies beyond the scope of the present work. We shall give only a few estimates of the structure of the magnetic field in the stochastic region, which may be needed in future studies of plasma instabilities.

There are at least three factors contributing to the suppression of plasma instabilities in the stochastic region:

1. Neighboring field lines diverge exponentially: $\Delta(z) \sim \exp(hz/L)$ along the direction of the eigenvector of the linearized transformation (2) with $\lambda > 1$. The angle of this direction with the azimuth $\varphi$ is equal to $\theta_p \approx \xi/\Omega_\phi^2$ and depends only weakly on $\psi$ ($\Delta\theta_p \sim \theta_p/K$). The rate of divergence is determined by the estimate

$$h = |\Delta_{N+1}/\Delta_N| \approx \Omega_\phi^2 |\cos\psi| \sim \Omega_\phi^2. \tag{15}$$

2. At a sufficiently large distance between the lines ($\Delta \gtrsim r$), they begin to "mix." This leads to exponential relaxation of any inhomogeneity in $\varphi$ and of inhomogeneity in $r$ with size $\Delta r/r \lesssim \xi$, with the characteristic "time"

(length) of relaxation $\sim L/h$. At large distances (in $r$) diffusion occurs with coefficient $D \sim (\xi r)^2/L$ (11), which is equivalent to damping over a length $\sim L/(\xi k_\perp r)^2$; $k_\perp$ is the radial projection of the wave vector of the plasma perturbation.

3. As a result of relaxation and diffusion in $r$, the density and temperature gradients of the unperturbed plasma may be substantially reduced.

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*Note: Figure translations are in progress. See original paper for figures.*