DUALITY PRINCIPLES FOR BOUNDARY-VALUE PROBLEMS

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Abstract

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DUALITY PRINCIPLES FOR BOUNDARY-VALUE PROBLEMS

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1. Consider the system of differential equations

$$\mathcal{L}y = f(x, y, y', \dots, y^{(n-1)}), \tag{1}$$

where $y = \{y_1, \dots, y_m\}$, and $\mathcal{L}$ is the differential operator

$$\mathcal{L}y = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y \tag{2}$$

with continuous (for simplicity) matrix coefficients $a_i(x)$. Everywhere in what follows it is assumed that each initial value

$$y(a) = y_0, \quad y'(a) = y_0', \dots, y^{(n-1)}(a) = y_0^{n-1} \tag{3}$$

determines a unique solution

$$y = y(x; y_0, y_0', \dots, y_0^{n-1}) \tag{4}$$

of equation (1), defined on the interval $[a, b]$ (which in what follows is regarded as fixed).

We shall be interested in solutions of equation (1) satisfying boundary conditions of the general form

$$l(y) = 0, \tag{5}$$

where

$$l(y) = \{l_1(y), \dots, l_{mn}(y)\}, \tag{6}$$

and each $l_i(y)$ is some functional. It is natural to seek solutions of equation (1) in the space C^{n-1} of vector functions defined on $[a, b]$ with values in an m-dimensional space, continuously differentiable $n-1$ times. We shall assume that the functionals $l_i(y)$ are continuous on C^{n-1}; we emphasize that the functionals $l_i(y)$ need not be linear.

Each point

$$z_0 = \{y^0, y_0', \dots, y_0^{n-1}\} \tag{7}$$

of the mn-dimensional space R^{mn} will be regarded as an initial value (3). The solution (4) corresponding to this initial value defines an operator

$$Qz_0 = y(x; y_0, y_0', \dots, y_0^{n-1}), \tag{8}$$

acting from R^{mn} into C^{n-1}. Denote by U the operator

$$Uz = l(Qz) + z, \tag{9}$$

acting in R^{mn}. We shall call U the **shift operator**, since it becomes the ordinary shift along trajectories of differential equations in the case of systems of first-order equations with periodic boundary conditions ($y(b) = y(a)$).

The shift operator is continuous in R^{mn}. Its fixed points coincide with the initial values of solutions of system (1) satisfying the boundary conditions (5). It will be convenient for us to say that the indicated initial—

values coincide with the zeros of the continuous finite-dimensional vector field

$$\Phi(z) = z - Uz \quad (z \in R^{mn}). \tag{10}$$

2. Denote by P the linear operator that assigns to each vector-function $y(x) \in C^{n-1}$ the vector (7), composed of the initial conditions (3); P acts from C^{n-1} into R^{mn}. Denote by V the operator that assigns to each point (7) the solution of the homogeneous equation

$$\mathcal{L}y = 0,$$

satisfying the initial condition (3).

We pass to the nonhomogeneous equation

$$\mathcal{L}y = f(x)$$

with continuous right-hand side. As is known (see, for example, ([1])), the solution $y(x)$ of this equation satisfying the initial conditions (3) can be written in the form

$$y(x) = Vz_0 + \int_a^x K(x,t) f(t)\,dt, \tag{11}$$

where $K(x,t)$ is the so-called Cauchy matrix.

The following assertion is almost obvious:

Theorem 1. *The nonlinear operator*

$$Ay(x) = V[l(y) + Py] + \int_a^x K(x,t) f[t, y(t), y'(t), \dots, y^{(n-1)}(t)]\,dt \tag{12}$$

is completely continuous in C^{n-1}; its fixed points coincide with the solutions of equation (1) satisfying the boundary conditions (5).

By virtue of Theorem 1 the vector field

$$\Psi(y) = y - Ay \quad (y \in C^{n-1}) \tag{13}$$

is completely continuous, and its zeros coincide with the solutions of equation (1) satisfying the conditions (7).

3. Let G and Ω be bounded domains, respectively, in the finite-dimensional phase space R^{mn} and in the infinite-dimensional space C^{n-1} of vector-functions; denote their boundaries by $\dot{G}$ and $\dot{\Omega}$. We shall assume that on $\dot{G}$ there are no initial conditions of solutions of problem (1), (5), and on $\dot{\Omega}$ there are no solutions of this problem. Suppose, moreover, that the set of solutions of problem (1), (5) lying in Ω coincides with the set of solutions whose initial conditions lie in G. Then we shall say that the domains G and Ω have the same core.

The main result of the present article is the following general principle of duality.

Theorem 2. *Let the bounded domains G and Ω have the same core. Then the rotation $\gamma(\Phi; \dot{G})$ of the finite-dimensional vector field (10) on $\dot{G}$ coincides with the rotation $\gamma(\Psi; \dot{\Omega})$ of the completely continuous vector field (13) on $\dot{\Omega}$:*

$$\gamma(\Phi; \dot{G}) = \gamma(\Psi; \dot{\Omega}).$$

This theorem contains, as special cases, the duality principles proved in ([2,4]), connected with periodic solutions of ordinary differential equations and equations with deviating arguments.

4. Let us now consider the operator

$$A_1 y = QUPy \quad (y \in C^{n-1}), \tag{14}$$

where Q is the operator (8), U is the operator (9), and P is the operator defined at the beginning of § 2. The operator (14) is completely continuous in the space C^{n-1}. Solu-

solutions of problem (1), (5) obviously coincide with the zeros of the completely continuous vector field

$$\Psi_1(y) = y - A_1 y \qquad (y \in C^{n-1}). \tag{15}$$

Theorem 3. *Let the bounded domains G and Ω have the same core. Then the rotations on their boundaries of the fields (10) and (15) coincide:*

$$\gamma(\Phi; \dot{G}) = \gamma(\Psi_1; \dot{\Omega}).$$

The passage from problem (1), (5) to the equation

$$y = A_1 y$$

in the problem of periodic solutions of systems of first-order equations is equivalent to finding fixed points of the shift along trajectories of a differential equation in a Banach space (see (5)), to which the system under study can be reduced.

5. Problem (1), (5) can be reduced to finding fixed points of various operators different from (12) and (14). The rotations of the corresponding vector fields can also be related to the rotation of the finite-dimensional field (10). We restrict ourselves here to one (apparently the most important) case.

Suppose that the boundary conditions (5) are linear and homogeneous; suppose that the equation $\mathcal{L}y = 0$ has no nonzero solutions satisfying the boundary conditions (5). Then problem (1), (5) is equivalent to the nonlinear integral equation

$$y(x) = \int_a^b K_1(x,t) f[t, y(t), y'(t), \dots, y^{(n-1)}(t)]\, dt, \tag{16}$$

where $K_1(x,t)$ is the corresponding Green' s function. Consequently, the solutions of problem (1), (5) coincide with the zeros of the completely continuous vector field

$$\Psi_2(y) = y - A_2 y \qquad (y \in C^{n-1}), \tag{17}$$

where

$$A_1 y(x) = \int_a^b K_1(x,t) f[t, y(t), y'(t), \dots, y^{(n-1)}(t)]\, dt. \tag{18}$$

Let β be the sum of the multiplicities of the negative eigenvalues of the matrix of the linear transformation $l(\dot{V}z)$ of the space R^{mn}.

Theorem 4. *Let the bounded domains G and Ω have the same core. Then the rotations on their boundaries of the fields (10) and (17) are related by the equality*

$$\gamma(\Phi; \dot{G}) = (-1)^\beta \gamma(\Psi_2; \dot{\Omega}). \tag{19}$$

6. The duality principles formulated above can be applied, according to the scheme set forth in (3), to the proof of the existence of solutions of boundary-value problems with distributed arguments and in various classes of integro-differential equations.

In conclusion we note that duality principles analogous to Theorems 2–4 are also valid in those cases when problem (1), (5) is reduced to operator equations in spaces different from C^{n-1}.

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Note: Figure translations are in progress. See original paper for figures.