ON THE PROBLEM OF CALCULATING UNSTEADY OCEAN CURRENTS AND TIDES

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Abstract

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ON THE PROBLEM OF CALCULATING UNSTEADY OCEAN CURRENTS AND TIDES

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In ([1]) the problem of determining steady currents in the ocean was considered. We shall generalize this problem to the case of unsteady currents caused by wind, the static nonuniformity of atmospheric pressure, climatological factors, and the tide-generating force. In addition to vertical exchange of momentum, we shall take horizontal exchange into account. The coefficients of vertical exchange of momentum and of turbulent diffusion will be regarded as variable.

We write the initial system of equations, boundary conditions, and initial conditions in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \Omega v - \Omega' w = \\ = g\frac{\partial \zeta'}{\partial x} - \frac{g}{\rho_0}\int_0^z \frac{\partial \rho}{\partial x}\,dz + \frac{\partial Q}{\partial x} + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) + A_l\Delta u, \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \Omega u = g\frac{\partial \zeta'}{\partial y} - \frac{g}{\rho_0}\int_0^z \frac{\partial \rho}{\partial y}\,dz + \\ + \frac{\partial Q}{\partial y} + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right) + A_l\Delta v;$$

$$w = \int_z^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz; \quad (2)$$

$$\partial\zeta/\partial t = \partial S_x/\partial x + \partial S_y/\partial y; \quad (3)$$

$$\rho = f(\tau, s) + \delta z; \quad (4)$$

$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} + v\frac{\partial \tau}{\partial y} + w\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z}\left(\chi_{\tau z}\frac{\partial \tau}{\partial z}\right) + \frac{\partial}{\partial x}\left(\chi_{\tau x}\frac{\partial \tau}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi_{\tau y}\frac{\partial \tau}{\partial y}\right), \quad (5)$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\chi_{sz}\frac{\partial s}{\partial z}\right) + \frac{\partial}{\partial x}\left(\chi_{sx}\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi_{sy}\frac{\partial s}{\partial y}\right);$$

$$\zeta = \zeta' + \frac{1}{g\rho_0}p_a; \quad (6)$$

$$\iint_\sigma \zeta\, dx\, dy = 0; \quad (7)$$

for $z = 0$

$$A_z\frac{\partial u}{\partial z} = -\frac{T_x}{\rho_0}, \qquad A_z\frac{\partial v}{\partial z} = -\frac{T_y}{\rho_0}; \quad (8)$$

$$a_{0\tau}\frac{\partial \tau}{\partial z} + b_{0\tau}\tau = \Gamma_{0\tau}, \qquad a_{0s}\frac{\partial s}{\partial z} + b_{0s}s = \Gamma_{0s}; \quad (9)$$

for $z = H$

$$u = v = 0; \quad (10)$$

$$a_{H\tau}\frac{\partial \tau}{\partial z} + b_{H\tau}\tau = \Gamma_{H\tau}, \qquad a_{Hs}\frac{\partial s}{\partial z} + b_{Hs}s = \Gamma_{Hs}; \quad (11)$$

on the contour L

$$u = v = 0; \quad (12)$$

$$a_{L\tau}\frac{\partial \tau}{\partial n} + b_{L\tau}\tau = \Gamma_{L\tau}, \qquad a_{Ls}\frac{\partial s}{\partial z} + b_{Ls}s = \Gamma_{Ls}; \quad (13)$$

for $t = 0$

$$u = u_0, \qquad v = v_0; \quad (14)$$

$$\zeta = \zeta_0; \quad (15)$$

$$\tau = \tau_0, \qquad s = s_0, \tag{16}$$

where u, v, w are the components of the current velocity along the Cartesian coordinate axes X, Y, Z (the Z-axis is directed vertically downward; the origin of coordinates is chosen so that condition (7) is satisfied); t is time; g is the acceleration due to gravity; $\Omega = 2\omega \sin\varphi$ is the Coriolis parameter, in which ω is the angular velocity of the Earth' s rotation and φ is latitude; $\Omega' = 2\omega\cos\varphi$; ρ is density and ρ_0 is the mean density of seawater; τ is temperature; s is salinity; δ is the coefficient of compressibility of seawater; ζ is the level; p_a is atmospheric pressure; A_z and A_l are the coefficients of vertical and horizontal exchange of momentum; $\varkappa_\tau$ and $\varkappa_s$ are the coefficients of turbulent diffusion of heat and salt; T_x, T_y are the components of the tangential wind stress; Q is the potential of the tide-generating force; L is the contour of the closed basin and n is the direction of the normal to it; σ is the area of the basin; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. Finally, S_x and S_y are the components of the total transport, determined with sufficient accuracy by the formulas

$$S_x = \int_0^H u\,dz, \qquad S_y = \int_0^H v\,dz. \tag{17}$$

The function f, which establishes the relation between density, temperature, and salinity; the quantities a, b, Γ, connected with diffusion processes at the boundaries of the basin; and the functions $u_0, v_0, \zeta_0, \tau_0, s_0$, characterizing the initial state, are assumed known.

The problem under consideration can be solved numerically by the splitting method ([2]). We divide the time step $t_{n+1} - t_n$ into three equal parts and solve systems of equations: on the first part of the step,

$$\frac{1}{3}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = A_l\frac{\partial^2 u}{\partial x^2}, \qquad \frac{1}{3}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = A_l\frac{\partial^2 v}{\partial x^2}; \tag{18}$$

$$\frac{1}{3}\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} = \frac{\partial}{\partial x}\left(\varkappa_{\tau x}\frac{\partial \tau}{\partial x}\right), \qquad \frac{1}{3}\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\varkappa_{sx}\frac{\partial s}{\partial x}\right); \tag{19}$$

on the second,

$$\frac{1}{3}\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = A_l\frac{\partial^2 u}{\partial y^2}, \qquad \frac{1}{3}\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = A_l\frac{\partial^2 v}{\partial y^2}; \tag{20}$$

$$\frac{1}{3}\frac{\partial \tau}{\partial t} + v\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial y}\left(\varkappa_{\tau y}\frac{\partial \tau}{\partial y}\right), \qquad \frac{1}{3}\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\varkappa_{sy}\frac{\partial s}{\partial y}\right); \tag{21}$$

on the third

$$\frac{1}{3}\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial z} - \Omega v - \Omega' w = g\frac{\partial \zeta'}{\partial x} - \frac{g}{\rho_0}\int_0^z \frac{\partial \rho}{\partial x}\,dz + \frac{\partial Q}{\partial x} + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right), \quad (22)$$

$$\frac{1}{3}\frac{\partial v}{\partial t} + w\frac{\partial v}{\partial z} + \Omega u = g\frac{\partial \zeta'}{\partial y} - \frac{g}{\rho_0}\int_0^z \frac{\partial \rho}{\partial y}\,dz + \frac{\partial Q}{\partial y} + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right);$$

$$\frac{1}{3}\frac{\partial \tau}{\partial t} + w\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z}\left(\chi_{\tau z}\frac{\partial \tau}{\partial z}\right), \qquad \frac{1}{3}\frac{\partial s}{\partial t} + w\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\chi_{sz}\frac{\partial s}{\partial z}\right). \quad (23)$$

Equations (18)–(23) are approximated by an implicit difference scheme. The quantities u, v, w, not appearing under the signs of the derivatives $\partial\zeta'/\partial x$, $\partial\zeta'/\partial y$, $\partial\rho/\partial x$, $\partial\rho/\partial y$, $\Omega' w$, are taken at the instant $t = t_n$. The first derivatives in the nonlinear terms are replaced by one-sided differences, with account taken of the signs of their coefficients ([2–4])*. The resulting three-point difference equations are solved by the sweep method. In order to reduce the effect of numerical viscosity that arises in this case, one may evidently use known methods, for example a corrector scheme ([2]).

Having found, at the instant $t = t_{n+1}$, the quantities u, v, τ, s, we find at the same instant the values S_x, S_y, w, ρ from formulas (17), (2), (4). The quantity ζ is computed for $t = t_{n+1}$ from equation (3)**. Then, for $t = t_{n+1}$, we compute the values

$$\frac{\partial \zeta'}{\partial x}, \quad \frac{\partial \zeta'}{\partial y}, \quad \int_0^z \frac{\partial \rho}{\partial x}\,dz, \quad \int_0^z \frac{\partial \rho}{\partial y}\,dz \quad \text{and so on.}$$

If, in the equations of motion, the terms due to horizontal exchange are not taken into account, then instead of condition (12) one should adopt the softened condition ([5,1]): on the contour L, $S_n = 0$. In this case the proposed method can be used if the nonlinear inertial terms are neglected, or if part of them (the horizontal advection) is taken at the preceding time instant. The computation is then substantially simplified, since there is no need to split the equations of motion. If, moreover, one confines oneself to studying currents that vary slowly in time (in particular, in this case it is necessary to exclude tidal currents by putting $Q = 0$), then in equations (1) one may neglect the terms $\partial u/\partial t$ and $\partial v/\partial t$ ([6]). In this case one can use the analytical expressions for u and v obtained in solving the stationary problem ([1])***.

In the proposed method the level is determined by numerical integration of the continuity equation in the integral form (3). In the case of currents varying slowly in time, the problem can be reduced to solving a system of equations including an equation for the level, as A. S. Sarkisyan ([6]) does for a deep basin located outside the equator. However, the solution of this system of equations for arbitrary depths or in the presence of the equator is extremely difficult, which is connected with the complexity of the boundary conditions for the level.

In ([7]) a linear problem on tidal currents in an inhomogeneous fluid was considered. This problem need not be reduced to the solution of a system

* At the instant $t = t_n$ one may take the nonlinear terms in full. As a rule, however, the difference scheme then becomes less stable.

** To compute ζ one may use the equality

$$\frac{\partial \iint_\Sigma \zeta \, d\sigma}{\partial t} = -\iint_\Sigma \operatorname{div} \mathbf{S} \, d\sigma = \oint_l S_n \, dl,$$

where l is the closed contour bounding an arbitrary region Σ.

*** If $\partial u/\partial t$, $\partial v/\partial t$ are represented in the form $(u^{n+1} - u^n)/\Delta t$, $(v^{n+1} - v^n)/\Delta t$, and it is assumed that the equations of motion are valid at the instant $t = t_{n+1}$, then, just as in ([1]), they can be integrated and analytical expressions can be obtained for u^{n+1}, v^{n+1}.

elliptic equations (78) or (80) for the harmonic constants of the level, which is difficult to carry out because of the complexity of the boundary conditions (79) or (81). A method analogous to the one we propose makes it possible to overcome this difficulty. It consists in constructing an iterative process by means of formulas (59)–(62), (54)–(56), by which all functions inside the domain are determined. On the boundary of the domain the calculations are carried out by formulas (56) and conditions (57) are adopted. We note that the method can also be used when the coefficient A_z depends on the vertical coordinate. In this case, without using formulas (59)–(62), it is necessary to solve equations (34) numerically.

Within the framework of the methods proposed in (1) and in the present paper, it is possible, by successive approximations, to take into account the principal terms that are neglected when the hydrostatic condition is adopted. For this purpose it is sufficient to represent the horizontal pressure gradients in the form

$$\frac{\partial p}{\partial x} = -g\rho_0 \frac{\partial \zeta'}{\partial x} + g \int_0^z \frac{\partial \rho}{\partial x} \, dz + \frac{\partial p_1}{\partial x}, \qquad \frac{\partial p}{\partial y} = -g\rho_0 \frac{\partial \zeta'}{\partial y} + g \int_0^z \frac{\partial \rho}{\partial y} \, dz + \frac{\partial p_1}{\partial y}, \tag{24}$$

where

$$p_1 = \rho_0 \int_0^z \left[\frac{\partial}{\partial z} \left(A_z \frac{\partial w}{\partial z} \right) - u \frac{\partial w}{\partial x} - v \frac{\partial w}{\partial y} - w \frac{\partial w}{\partial z} - \Omega' u \right] dz, \tag{25}$$

and in concrete computations the value p_1 is to be taken at the preceding time instant.

We note that use of the proposed method, both in the general case and in special cases, is possible also when equations serving to determine the turbulence coefficients are added to the original system of equations, in particular the equation of the turbulent energy balance (8).

In conclusion, we point out that, in solving the problems under consideration, in addition to the splitting method one may use other methods, for example, solve the original system of equations by an explicit scheme.

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Note: Figure translations are in progress. See original paper for figures.