SovietRxiv

Soviet-era science, translated into English

ON THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS IN A NEIGHBORHOOD OF A HOMOCLINIC CURVE

MATHEMATICS

1967

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196701.37772

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

L. P. SHIL' NIKOV

ON THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS IN A NEIGHBORHOOD OF A HOMOCLINIC CURVE

(Presented by Academician L. S. Pontryagin, 11 III 1966)

§ 1. Consider a system of $m+n+1$ differential equations

$$\frac{dz}{dt} = Z(z), \tag{1}$$

where the right-hand sides are at least twice continuously differentiable in some domain $G \subset R^{n+m+1}$. Suppose that the system has a periodic motion $\mathfrak{L}_0$ of period 2π, whose characteristic exponents satisfy the following properties: except for one zero exponent, there are $m \geqslant 1$ exponents $\lambda_1, \dots, \lambda_m$ having negative real parts, and $n \geqslant 1$ exponents $\gamma_1, \dots, \gamma_n$ having positive real parts. As is known, in this case there exist two integral manifolds of dimensions $m+1$ and $n+1$, which intersect along $\mathfrak{L}_0$. We shall denote the stable integral manifold by $\mathfrak{M}^+$, and the unstable one by $\mathfrak{M}^-$.

Suppose that there exists a trajectory Γ_0, doubly asymptotic to $\mathfrak{L}_0$. Following Poincaré ([1]), who indicated the possibility of the existence of such motions (see also ([6])), we shall call Γ_0 a homoclinic curve. Since $\Gamma_0 \subset \mathfrak{M}^+ \cap \mathfrak{M}^-$, it is natural to assume with respect to this intersection that it is rough, i.e.,

$$\dim\left(W_M^+ \cap W_M^-\right) = 1, \tag{2}$$

where W_M^+ and W_M^- denote the tangent spaces to $\mathfrak{M}^+$ and $\mathfrak{M}^-$ at a point $M \in \Gamma_0$.

Theorem 1. *If condition (2) is fulfilled, then in every neighborhood of the homoclinic curve Γ_0 there is contained a countable number of periodic motions of saddle type.*

Remark 1. It follows from this theorem that the five well-known roughness conditions for dynamical systems, formulated by Smale ([3]) as hypotheses, are not independent.

SovietRxiv

Remark 2. As is known, a nonautonomous system

$$dz/dt = f(z, t)$$

with periodic right-hand side is reduced to the study of the autonomous system

$$\begin{aligned} dz/dt &= f(z, \theta), \\ d\theta/dt &= 1 \end{aligned}$$

with toroidal phase space, which has a global section.

Birkhoff ([2]) proved for the case of a point mapping of the plane into the plane preserving area that in a neighborhood of a homoclinic point there is contained a countable set of periodic points. However, Birkhoff' s method cannot be extended to the general case.

Our proof of Theorems 1 and 2 does not use the construction of a global section.

Remark 3. Other cases of the existence of a countable set of periodic motions connected with a nontransversal intersection of integral manifolds are considered in the works ([10–12]). A number of Smale' s works ([4,5]) are devoted to questions connected with the roughness of homeomorphisms having a countable set of periodic motions.

The investigation of the roughness and structure of a homoclinic cell is the subject of a work of Yu. I. Neimark and the author (see ([7])).

§ 2. The proof of Theorem 1 is based on the construction of a point mapping in a neighborhood of Γ_0, using the so-called parametric representation of the mapping in a neighborhood of $\mathfrak{L}_0$. (On the parametric representation of a mapping see also ([12]).)

With the help of a certain change of variables, system (1) in a neighborhood of $\mathfrak{L}_0$ can be brought to the form

$$\begin{aligned} dx/dt &= Ax + P(x, y, \theta)x, \\ dy/dt &= By + Q(x, y, \theta)y, \\ d\theta/dt &= 1, \end{aligned} \tag{3}$$

where A and B are constant matrices of orders m and n, reduced to normal Jordan form, whose characteristic roots are respectively $\lambda_1, \dots, \lambda_m$ and $\gamma_1, \dots, \gamma_n$; P and Q are periodic functions of θ, in the general case of period 4π, which vanish for $x = y = 0$. The equations of $\mathfrak{M}_+$ and $\mathfrak{M}_-$ in these variables have respectively the form

$$y = 0, \qquad x = 0. \tag{4}$$

SovietRxiv

Denote by S_0 the surface

$$\|x\|^2 = r^2, \qquad \|y\|^2 < r^2,$$

and by S_1

$$\|y\|^2 = r^2, \qquad \|x\|^2 < r^2.$$

Lemma 1. *For all sufficiently small $r > 0$ and $0 \le \theta \le 4\pi$, the surfaces S_0 and S_1 are surfaces without contact for the trajectories of system (3).*

It follows from Lemma 1 that a trajectory l, passing through the point $M_0(x_0, y_0, \theta_0) \in S_0$, where $\|y_0\| \ne 0$, after a time t_0 will intersect S_1 at some point $M_1(x_1, y_1, \theta_1)$. Denote this mapping by T_0.

Let

$$\begin{aligned} x(t) &= x(t, x_0, y_0, \theta_0), \\ y(t) &= y(t, x_0, y_0, \theta_0), \\ \theta(t) &= t + \theta_0 \end{aligned}$$

be the equations of l. Then T_0 can be written in the form

$$\begin{aligned} x_1 &= x(t_0, x_0, y_0, \theta_0), \\ y_1 &= y(t_0, x_0, y_0, \theta_0), \\ \theta_1 &= t_0 + \theta_0 \mod \tau \quad (\tau \text{ is equal either to } 2\pi \text{ or to } 4\pi), \end{aligned} \tag{5}$$

where the transition time t_0 is determined from the equation

$$\|y(t_0, x_0, y_0, \theta_0)\|^2 = r^2.$$

It is not difficult to prove that the mapping T_0 can be written in the form

$$\begin{aligned} x_1 &= x^{\mathrm{p}}(t_0, x_0, y_1, \theta_0), \\ y_1 &= y^{\mathrm{p}}(t_0, x_0, y_1, \theta_0), \\ \theta_1 &= t_0 + \theta_0 \mod \tau, \qquad \|x_0\|^2 = r^2, \quad \|y_1\|^2 = r^2, \end{aligned} \tag{6}$$

where x^{p} and y^{p} tend to zero as $t_0 \to \infty$, together with all first derivatives. In the parametric form (6), the mapping T_0 is defined for all x_0, y_1 satisfying the condition $\|x_0\|^2 = r^2$, $\|y_1\|^2 = r^2$, and $t_0 > 0$.

Let $M_0^+(x_0^+, 0, \theta_0^+)$ and $M_1^-(0, y_1^-, \theta_1^-)$ be the points of intersection of Γ_0 with S_0, S_1, and let U_0, U_1 be their neighborhoods on S_0, S_1:

$$U_0 = [M_0(x_0, y_0, \theta_0) : \ \|x_0 - x_0^+\| < \varepsilon, \ \|y_0\| < \varepsilon, \ |\theta_0 - \theta_0^+| < \varepsilon],$$

SovietRxiv

$$U_1 = [M_1(x_1, y_1, \theta_1): \ \|x_1\| < \delta, \ \|y_1 - y_1^-\| < \delta, \ |\theta_1 - \theta_1^-| < \delta].$$

Let ε and δ be such that the vectors x_0 and y_1 can be represented in the form $x_0 = (\tilde{x}_0, x_0')$, $y_1 = (\tilde{y}_1, y_1')$, where $\tilde{x}_0$ is expressed through the components of the $(m-1)$-dimensional vector x_0', and $\tilde{y}_1$ through the components of the $(n-1)$-dimensional vector y_1'.

From the theorem on continuous dependence on initial conditions it follows that, along trajectories close to Γ_0, one can establish a one-to-one correspondence between certain neighborhoods of the points M_0^+ and M_1^-. Denote this mapping by T_0. Let δ be such that $T_1 U_1 \subset U_0$. The mapping T_1 in the variables under consideration can be written in the form

$$\bar{x}_0' = F(x_1, y_1', \theta_1),$$

$$\bar{y}_0 = G(x_1, y_1', \theta_1),$$
$$\bar{\theta}_0 = \Phi(x_1, y_1', \theta_1),$$

or

$$\Delta\bar{x}_0' = A_1 x_1 + B_1 \Delta y_1' + C_1 \Delta\theta_1 + \dots,$$
$$\bar{y}_0 = A_2 x_1 + B_2 \Delta y_1' + C_2 \Delta\theta_1 + \dots,$$
$$\Delta\bar{\theta}_0 = a x_1 + b \Delta y_1' + c \Delta\theta_1 + \dots, \tag{8}$$

where $\Delta\bar{x}_0' = \bar{x}_0' - (x_0^+)'$, $\Delta y_1' = y_1' - (y_1^-)'$, $\Delta\bar{\theta}_0 = \bar{\theta}_0 - \theta_0^+$, $\Delta\theta_1 = \theta_1 - \theta_1^-$.

Fulfillment of condition (2) means that

$$\operatorname{rang} \|B_2 C_2\| = n. \tag{9}$$

From Lemma 1, (9), and the parametric representation (6) of the mapping T_0, it follows that $\bar{y}_0 \neq 0$ can be represented in the form

$$\bar{y}_0 = y^{\Pi}(\bar{t}_0, \bar{x}_0, \bar{y}_1, \bar{\theta}_0), \tag{10}$$

where $\bar{t}_0$ is the time of passage of the phase point from the point $\bar{M}_0(\bar{x}_0, \bar{y}_0, \bar{\theta}_0) \in S_0$ to the point $\bar{M}_1(\bar{x}_1, \bar{y}_1, \bar{\theta}_1) \in S_1$.

Consider the mapping $T = T_1 T_0$. It can be written in the form

$$\bar{x}_0' = F(x^{\Pi}(t_0, x_0, y_1, \theta_0),\, Y_1',\, t_0 + \bar{\theta}_0),$$

$$y^{\Pi}(\bar{t}_0, \bar{x}_0, \bar{y}_1, \bar{\theta}_0) = G(x^{\Pi}(t_0, x_0, y_1, \theta_0),\, Y_1',\, t_0 + \bar{\theta}_0),$$
$$\bar{\theta}_0 = \Phi(x^{\Pi}(t_0, x_0, y_1, \theta_0),\, y_1',\, t_0 + \bar{\theta}_0). \tag{11}$$

It is easy to see that it is defined for $\|\Delta x_0'\| < \varepsilon$, $\|\Delta y_1'\| < \delta$, $|\theta_0 - \theta_0^+| < \varepsilon$, $|t_0 + \theta_0 - \theta_1^-| < \delta \pmod{4\pi}$, and maps the point $(x_0', y_1', t_0, \theta_0)$ to the point $(\bar{x}_0', \bar{y}_1', \bar{t}_0, \bar{\theta}_0)$. Representing t_0 in the form $t_0 = \tau k - \theta_0^+ + \theta_1^- + \tau_0$, we obtain

SovietRxiv

that the domain of definition of T consists of a countable union of disjoint domains

$$\sigma_k = [\|\Delta x_0'\| < \varepsilon,\ \|\Delta y_1'\| < \delta,\ |\theta_0 - \theta_0^+| < \varepsilon,\ |\tau_0 + \Delta\theta_0| < \delta],$$

where $k = \bar{k}, \bar{k} + 1, \ldots$.

Thus the problem of the existence of a countable set of periodic motions in $U(\Gamma_0)$ has been reduced to the problem of finding fixed points of the mapping T and of its powers.

Lemma 2. *The mapping T has a countable number of fixed points of saddle type.*

The fixed points of the mapping T are found from the system

$$\begin{aligned} f &\equiv x_0' - F(x^\Pi, y_1', \theta_0 + \tau_0 + \theta_0^+) = 0, \\ g &\equiv y^\Pi - G(x^\Pi, y_1', \theta_0 + \tau_0 + \theta_0^+) = 0, \\ \varphi &\equiv \theta_0 - \Phi(x^\Pi, y_1', \theta_0 + \tau_0 + \theta_0^+) = 0. \end{aligned} \tag{12}$$

Using the condition

$$\lim \frac{D(f, g, \varphi)}{D(x_0', y_1', \theta_0, \tau_0)} = (-1)^{n-1}|B_2 C_2| \neq 0, \tag{13}$$

we easily prove that the mapping T in each σ_k, for $k \geq \bar{k}' \geq \bar{k}$, has a fixed point M_k^*.

Linearizing (11) in a neighborhood of an equilibrium point and forming the characteristic equation, we are convinced that each equilibrium point is a rough point of saddle type.

Let N_p be the set of equilibrium points of the mapping T of multiplicity p, i.e. equilibrium points of the mapping T^p. In an analogous manner it is shown that N_p is nonempty for all $p \geqslant 2$.

From the construction of the mapping T it follows that in any neighborhood of the homoclinic curve Γ_0 there is contained a countable set of periodic motions.

§ 3. Suppose that system (1) has periodic motions $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_k$ of saddle type and trajectories $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$, where $l \geqslant k$, such that $\alpha(\Gamma_i) = \omega(\Gamma_{i+1})$, $\alpha(\Gamma_0) = \omega(\Gamma_l)$, where by $\alpha(\Gamma_i)$ and $\omega(\Gamma_i)$ are denoted the α- and ω-limit sets of the trajectory Γ_i, which are the indicated periodic solutions. Let $U(\Gamma_1, \ldots, \Gamma_l)$ be a certain neighborhood of the contour $\Gamma_1, \ldots, \Gamma_l$, which we shall call a homoclinic contour.

Analogously to Theorem 1, one proves

Theorem 2. *If the stable and unstable manifolds of the periodic motions intersect along Γ_i roughly, then in any neighborhood of the homoclinic contour*

SovietRxiv

$U(\Gamma_1, \ldots, \Gamma_l)$ *there is contained a countable set of periodic motions of saddle type.*

Here the proof reduces to the investigation of the fixed points of the mapping T, representable in the form of a product of mappings $T_1^{(i)} T_0^{(i)}$, where $T_0^{(i)}$ is the mapping in a neighborhood of $\mathfrak{L}_i$, which is constructed analogously to the mapping T_0 of § 2, and $T_1^{(i)}$ is the mapping in a neighborhood of Γ_i, which is constructed analogously to the mapping T_1.

Remark. The study of the mapping T makes it possible to establish the existence of an invariant set analogous to that described by Smale (13).

Scientific Research Institute
of Applied Mathematics and Cybernetics
at the Gorky State University
named after N. I. Lobachevsky

Received
10 III 1966

CITED LITERATURE

1. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, 1-3, Paris, 1899.
2. G. D. Birkhoff, *Nouvelles recherches sur les systèmes dynamiques*, *Memoriale Pont. Acad. Novi Lyncaei*, 1935, s. 3, 1.
3. S. Smale, *Bull. Am. Math. Soc.*, 66, 43 (1960).
4. S. Smale, *Tr. Mezhdunarodn. simpoziuma po nelineinym kolebaniyam*, Kiev, 1963.
5. S. Smale, *International Congress of Mathematicians at Stockholm*, 1962.
6. V. K. Melnikov, *Tr. Mosk. matem. obshch.*, 12, 1963.
7. Yu. I. Neimark, *Tr. II Vsesoyuzn. syezda po teoret. i prikl. mekh.*, 2, 1965, p. 97.
8. L. P. Shilnikov, *DAN*, 143, No. 2, 289 (1962).
9. L. P. Shilnikov, *Matem. sborn.*, 61, (104), 4, 443 (1963).
10. L. P. Shilnikov, *DAN*, 162, No. 3, 558 (1965).
11. Yu. I. Neimark, L. P. Shilnikov, *DAN*, 162, No. 6, 1261 (1965).

SovietRxiv

12. L. P. Shilnikov, *DAN*, 172, No. 1 (1967).

13. S. Smale, *Diffeomorphisms with Many Periodic Points*, *Differential and Combinatorial Topology*, Princeton Mathematical Series, No. 27, 1965.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.