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Abstract

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MATHEMATICS

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ON THE STABILITY OF THE ROOT NUMBER OF AN ANALYTIC OPERATOR-FUNCTION AND ON PERTURBATIONS OF ITS CHARACTERISTIC NUMBERS AND EIGENVECTORS

(Presented by Academician I. N. Vekua on 9 VI 1966)

In the present note, the theorem of I. Ts. Gohberg and M. G. Krein (([1]), Theorem 4.3) on the stability of the root number of a linear operator and the theorem on analytic perturbations of characteristic numbers and eigenvectors of a linear operator (see, for example, ([2])) are extended to the case of an analytic operator-function.

1. Let G be some domain in the complex plane, and let $A(\lambda)$ be an operator-function analytic in the domain G, whose values are bounded linear operators acting in a Banach space $\mathfrak{B}$. A number $\lambda_0 \in G$ is called a **characteristic number** of the operator-function $A(\lambda)$ if the equation $A(\lambda_0)\varphi = 0$ has a nontrivial solution φ_0, and the vector φ_0 is called an **eigenvector** of the operator-function $A(\lambda)$ corresponding to λ_0. The number $a(A(\lambda_0)) = \dim \mathfrak{Z}(A(\lambda_0))$, where $\mathfrak{Z}(A(\lambda_0))$ is the subspace of all eigenvectors corresponding to λ_0, will be called the **geometric multiplicity** of the characteristic number λ_0. A set of vectors $\varphi_1, \varphi_2, \dots, \varphi_k$ is called a **chain of vectors associated** with the eigenvector φ_0 of the operator-function $A(\lambda)$, corresponding to the number λ_0, if

$$\sum_{q=0}^{j} \frac{1}{q!} A^{(q)}(\lambda_0)\varphi_{j-q} = 0 \qquad (j = 1, 2, \dots, k).$$

The number $k+1$ is called the **length** of this chain. By the multiplicity of the eigenvector φ_0 we shall mean the number $m(\varphi_0)$, equal to the maximal length of a chain of vectors associated with φ_0. A **canonical system of eigenvectors and associated vectors** of the operator-function $A(\lambda)$, corresponding to the number λ_0, is a system $\varphi_{j0}, \varphi_{j1}, \dots, \varphi_{j,m_j-1}$ $(j = 1, 2, \dots, \alpha = \alpha(A(\lambda_0)))$, where φ_{j0} is an eigenvector of multiplicity m_j, and $\varphi_{j1}, \varphi_{j2}, \dots, \varphi_{j,m_j-1}$ is a chain of

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vectors associated with it, with m_1 being the maximal of the multiplicities of all eigenvectors corresponding to λ_0, and m_j $(j = 2, 3, \dots, \alpha)$ being the maximal of the multiplicities of all eigenvectors from some direct complement in $\mathfrak{Z}(A(\lambda_0))$ to the subspace spanned by $\varphi_{10}, \dots, \varphi_{j-1,0}$. The number

$$m(\lambda_0, A) = \sum_{j=1}^{\alpha} m_j$$

is called the **algebraic multiplicity** of the characteristic number λ_0 of the operator-function $A(\lambda)$. A point λ will be called a **regular point** of the operator-function $A(\lambda)$ if the operator $A(\lambda)$ is continuously invertible, and a **Φ-point** of the operator-function $A(\lambda)$ if the operator $A(\lambda)$ is a Φ-operator (see ([1]), p. 52).

Let Γ be a closed rectifiable contour situated in the domain G and having, with respect to the operator-function $A(\lambda)$, the following properties:

a) inside the contour Γ there is a finite number of characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_s$ of the operator function $A(\lambda)$, which are its Φ-points;

b) all the remaining points λ $(\lambda \neq \lambda_j,\ j = 1, 2, \dots, s)$ situated inside the contour and on the contour Γ are regular points of the operator function $A(\lambda)$.

From Theorem 1 of A. S. Markus ([3]) it follows that $m(\lambda_j, A) < \infty$ $(j = 1, 2, \dots, s)$. The **root number** of the operator function $A(\lambda)$, **corresponding to the contour** Γ, will mean the sum of the algebraic multiplicities of all characteristic numbers λ_j $(j = 1, 2, \dots, s)$ of the operator function $A(\lambda)$ located inside the contour Γ, i.e. the number

$$m(A, \Gamma) = m(\lambda_1, \Gamma) + m(\lambda_2, \Gamma) + \dots + m(\lambda_s, A).$$

Theorem 1. *Let Γ be a closed rectifiable contour bounding a domain G, and let $A(\lambda)$ be an operator function, analytic in G and continuous in the closed domain $G \cup \Gamma$, with the contour Γ possessing, relative to the operator function $A(\lambda)$, properties a), b). Then there exists a number $\delta > 0$ such that, whatever the operator function $B(\lambda)$, analytic in G, continuous in $G \cup \Gamma$, and satisfying the condition*

$$\max_{\lambda \in \Gamma} \|B(\lambda) - A(\lambda)\| < \delta, \tag{1}$$

the contour Γ possesses properties a), b) also relative to $B(\lambda)$, and moreover

$$m(B, \Gamma) = m(A, \Gamma).$$

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For the proof of Theorem 1 the following two lemmas are needed:

Lemma 1. *Let* $\{A_j\}_0^\infty$ *be a sequence of linear bounded operators acting in a Banach space* $\mathfrak{B}$ *and satisfying the condition* $\sup \|A_j\| < \infty$, *and*

$$A(\lambda) = \sum_{j=0}^{\infty} \lambda^j A_j \qquad (|\lambda| < 1).$$

Let, further, $\widetilde{\mathfrak{B}}$ *be the Banach space of all sequences* $\tilde{x} = \{x_j\}_1^\infty$ *of elements from* $\mathfrak{B}$ *satisfying the condition*

$$\|\tilde{x}\| = \sum_{j=1}^{\infty} \|x_j\| < \infty,$$

and let $\widetilde{A}_0, \widetilde{C}$ *be the operators defined in* $\widetilde{\mathfrak{B}}$ *by the matrices*

$$\widetilde{A}_0 = \begin{pmatrix} A_0 & A_1 & \cdots & A_n & \cdots \\ 0 & I & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots \\ -I & 0 & \cdots & 0 & \cdots \\ 0 & -I & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

A set of vectors $\widetilde{\varphi}_j = \{\varphi_j^{(k)}\}_{k=0}^\infty$ $(j = 0, 1, \dots, p)$ *is then and only then a chain of eigenvectors and associated vectors of the linear pencil* $\widetilde{A}(\lambda) = \widetilde{A}_0 + \lambda\widetilde{C}$, *corresponding to the characteristic number* λ $(|\lambda| < 1)$, *when* $\varphi_j^{(0)}$ $(j = 0, 1, \dots, p)$ *is a chain of eigenvectors and associated vectors of the operator function* $A(\lambda)$, *corresponding to the characteristic number* λ, *and*

$$\varphi_0^{(k)} = \lambda\varphi_0^{(k-1)} \qquad (k = 1, 2, \dots, \infty); \quad \varphi_j^{(k)} = \lambda\varphi_j^{(k-1)} + \varphi_{j-1}^{(k-1)}$$

$$(j = 1, 2, \dots, p;\ k = 1, 2, \dots, \infty).$$

Lemma 2. Let $A(\lambda)$ and $B(\lambda)$ be operator-functions, analytic inside the circle $|\lambda| < r$ and continuous in the closed circle $|\lambda| \le r$, and let Γ be a closed rectifiable contour lying inside the circle $|\lambda| < r$. If the contour Γ has, with respect to the operator-function $A(\lambda)$, properties a), b), then there exists a number $\delta > 0$ such that, when the condition

$$\max_{|\lambda|=r} \|B(\lambda) - A(\lambda)\| < \delta$$

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is satisfied, the contour Γ has properties a), b) also with respect to the operator-function $B(\lambda)$, and moreover

$$m(B, \Gamma) = m(A, \Gamma).$$

Lemma 1 is a generalization of a well-known property of polynomial operator pencils (see (4), p. 326). Lemma 2 is proved on the basis of Lemma 1 and the theorem from (5).

Proof of Theorem 1. Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be all the characteristic numbers of the operator-function $A(\lambda)$ in the domain G (without regard to multiplicity). Choose numbers $\varepsilon_j > 0$ ($j = 1, 2, \dots, s$) such that the disks $G_j = \{\lambda : |\lambda - \lambda_j| < \varepsilon_j\}$ lie in G and are pairwise disjoint. Put $\gamma_j = \{\lambda : |\lambda - \lambda_j| = \varepsilon_j/2\}$. By Lemma 2, for each $j = 1, 2, \dots, s$ there exists a number $\delta_j > 0$ such that from the inequality

$$\max_{|\lambda - \lambda_j| = \varepsilon_j} \|B(\lambda) - A(\lambda)\| < \delta_j \tag{2}$$

it follows that the contours γ_j have, with respect to $B(\lambda)$, properties a), b), and moreover

$$m(B, \gamma_j) = m(A, \gamma_j). \tag{3}$$

Denote by F the closed set obtained by removing from $G \cup \Gamma$ the open disks $|\lambda - \lambda_j| < \varepsilon_j/2$ ($j = 1, 2, \dots, s$). For $\lambda \in F$ the operator $A(\lambda)$ is invertible. Put

$$M = \max_{\lambda \in F} \|A^{-1}(\lambda)\|.$$

We now show that the number

$$\delta = \min(\delta_1, \delta_2, \dots, \delta_s, 1/M)$$

has the required property. Indeed, if inequality (1) is satisfied, then for any point $\lambda \in F$

$$\|B(\lambda) - A(\lambda)\| < \delta \le 1/M \le 1/\|A^{-1}(\lambda)\|,$$

and therefore the operator $B(\lambda)$ is invertible.

On the other hand, from (1) it follows that, for all $j = 1, 2, \dots, s$, (2) holds, and therefore all contours γ_j have, with respect to $B(\lambda)$, properties a), b), and the equalities (3) hold.

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From what has been said above it follows directly that the contour Γ has properties a), b) with respect to the operator-function $B(\lambda)$ and

$$m(B,\Gamma)=\sum_{j=1}^{s} m(B,\gamma_j)=\sum_{j=1}^{s} m(A,\gamma_j)=m(A,\Gamma).$$

The theorem is proved.

2. Theorem 2. Let $\{A_j(\varepsilon)\}_0^\infty$ be a sequence of operator-functions, analytic in some disk $|\varepsilon|<\delta$ and satisfying the condition

$$\sup_j \|A_j(\varepsilon)\|<\infty \qquad (|\varepsilon|<\delta);$$

$$A_\varepsilon(\lambda)=\sum_{j=0}^{\infty} A_j(\varepsilon)\lambda^j \qquad (|\lambda|<1,\ |\varepsilon|<\delta).$$

Let Γ be a closed rectifiable contour lying in the circle $|\lambda|<1$, inside which there is a single characteristic number λ_0 of the operator-function $A_0(\lambda)$, of finite algebraic multiplicity m. Then:

1) There exists a number ρ $(0<\rho\le\delta)$ and a natural number r $(\le m)$ such that, for $0<|\varepsilon|<\rho$, the operator-function $A_\varepsilon(\lambda)$ has exactly r distinct characteristic numbers inside the contour Γ. These characteristic numbers can be arranged into groups $\lambda_{ij}(\varepsilon)$ $(i=1,2,\dots,l;\ j=1,2,\dots,p_i;\ \sum_{i=1}^{l} p_i=r)$ in such a way that the functions of one group (i.e.

$$\lambda_{i1},\lambda_{i2},\dots,\lambda_{ip_i})$$

constitute the complete set of branches of a p_i-valued function; moreover, in the circle $|\varepsilon|<\rho$ they are represented in the form of series

$$\lambda_{ij}(\varepsilon)=\lambda_0+\sum_{k=1}^{\infty} a_{ik}\left[(\varepsilon^{1/p_i})_j\right]^k \quad (j=1,2,\dots,p_i;\ i=1,2,\dots,l),$$

where $(\varepsilon^{1/p_i})_j$ $(j=1,2,\dots,p_i)$ is the complete set of branches of the function ε^{1/p_i}.

2) There are numbers m_i $(i=1,2,\dots,l)$ such that $m(\lambda_{ij}(\varepsilon),A_\varepsilon)=m_i$ $(j=1,2,\dots,p_i;\ 0<|\varepsilon|<\rho)$, and

$$\sum_{i=1}^{l}\sum_{j=1}^{p_i} m(\lambda_{ij}(\varepsilon),A_\varepsilon)=\sum_{i=1}^{\infty} p_i m_i=m.$$

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3) There are numbers $\alpha_i \leq \alpha(A_0(\lambda_0))$ $(i = 1, 2, \dots, l)$ such that

$$\alpha(A_\varepsilon(\lambda_{ij})) = \alpha_i \quad (j = 1, 2, \dots, p_i;\ 0 < |\varepsilon| < \rho).$$

4) A basis of the eigenspace $\mathfrak{Z}(A_\varepsilon(\lambda_{ij}(\varepsilon)))$ can be represented, for $0 < |\varepsilon| < \rho$, in the form of series

$$\varphi_{ij}^{(q)}(\varepsilon) = \sum_{k=0}^{\infty} \varphi_{ik}^{(q)} \left[(\varepsilon^{1/p_i})_j\right]^k \quad (q = 1, 2, \dots, \alpha_i),$$

where $\varphi_{ik}^{(q)}$ are certain vectors, and $\varphi_{i0}^{(q)} \in \mathfrak{Z}(A_0(\lambda_0))$.

The proof of this theorem is carried out analogously to the proof of Theorem 2 of [6], with the aid of Lemma 1.

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Note: Figure translations are in progress. See original paper for figures.

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