Partial Differential Equation with Retarded Argument

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Abstract

A solution to the equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} - b^2 \frac{\partial^2 u(t-\tau,x)}{\partial t^2} \tag{1}$$

for $t \geq \tau$ is sought, satisfying zero boundary conditions

$$u(t,0) \equiv 0, \quad u(t,l) \equiv 0 \tag{2}$$

and initial conditions

$$u(t,x) = \varphi(t,x), \quad \frac{\partial}{\partial t} u(t,x) = \frac{\partial}{\partial t}\varphi(t,x) \quad \text{for} \quad 0 \leq t \leq \tau, \quad 0 \leq x \leq l.$$

Conditions have been found under which the solution to the problem can be represented as a series containing solutions to equation (1) corresponding to the initial functions 1, t, t^2. It is established that the presence of a time delay in the highest derivative leads to a loss of solution smoothness as the time interval expands, similar to the case for ordinary differential equations with a leading argument.

Bibliography: 5 items.

Full Text

Preamble

This section addresses the solution of a specific class of partial differential equations with time delay. We consider the following equation:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} - b^2 \frac{\partial^2 u(t-\tau,x)}{\partial x^2} + f(t,x) \tag{1}$$

where $a^2 > b^2 > 0$ and $\tau > 0$ represents the time delay. The system is subject to the following boundary conditions:

$$u(t,0) = 0, \quad u(t,l) = 0 \tag{2}$$

To solve this problem, we employ the method of separation of variables by assuming a solution of the form $u(t,x) = T(t)X(x)$. Substituting this into the homogeneous part of equation (1), we obtain the spatial eigenvalue problem:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0 \tag{3}$$

The eigenvalues and corresponding eigenfunctions are given by:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad (n = 1, 2, \dots) \tag{4}$$

The general solution can be represented as a series:

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l} \tag{5}$$

Substituting (5) into (1), we derive a sequence of ordinary differential equations for the time-dependent coefficients $T_n(t)$:

$$T_n''(t) + a^2\lambda_n T_n(t) - b^2\lambda_n T_n(t-\tau) = f_n(t) \tag{6}$$

where $f_n(t)$ are the Fourier coefficients of the source term $f(t,x)$. For the initial interval $0 \le t \le \tau$, the function $T_n(t)$ is determined by the initial conditions. Let $B_n(t)$ represent the source term contribution. Using the method of successive integration over intervals of length τ, we can express the solution for $T_n(t)$ as:

$$T_n(t) = C_{0n} + C_{1n}t + C_{2n}t^2 + \int_0^t T_n(t-s)y_n(s)ds \tag{7}$$

The coefficients C_{0n}, C_{1n}, C_{2n} are determined by the continuity requirements at the boundaries of the time intervals. Specifically, for $t \in [k\tau, (k+1)\tau]$, the solution depends on the values of the function in the preceding interval $[(k-1)\tau, k\tau]$. By applying the properties of the integral kernel and the specific form of $B_n(t)$, we obtain:

$$C_{0n} = T_n(0) + \frac{\tau}{2}T_n'(\tau) - \dots \tag{8}$$

The final solution for the displacement $u(t,x)$ is constructed by summing the components:

$$u(t,x) = \sum_{n=1}^{\infty} \left\{ C_{0n} T_n(t) + C_{1n} T_n(t) + \int_0^t T_n(t-s) \left[h_n(s) - \int_0^s B_n(\xi) \cos\omega(s-\xi) d\xi \right] ds \right\} \sin\frac{n\pi x}{l} \tag{9}$$

Estimates for the coefficients and the integral terms show that for $t \in [k\tau, (k+1)\tau]$, the following inequality holds:

$$|T_n(t)|, |T_n'(t)|, |T_n''(t)| < n^{k-1} C_k \tag{10}$$

where C_k is a constant depending on the interval index. These bounds ensure the convergence of the series (9) and its derivatives, provided the source function $f(t,x)$ and the initial data possess sufficient smoothness (typically up to order $k+4$).

As a practical example, consider the case where $l = 1, \tau = 1$, and the source term is given by $f(t,x) = x^6(x-1)^6 \sin t$. The solution $u(t,x)$ can be computed numerically or analytically using the derived formulas for successive time intervals. The results demonstrate the influence of the delay parameter τ on the wave propagation characteristics within the medium.

Note: Figure translations are in progress. See original paper for figures.

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