ON THE QUESTION OF CONFORMAL MAPPING OF CIRCULAR DOMAINS

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Abstract

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MATHEMATICS

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ON THE QUESTION OF CONFORMAL MAPPING OF CIRCULAR DOMAINS

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1. Let L be a simple nonclosed curve composed of a finite number of arcs of circles and line segments, beginning at a point $\zeta_0 > 0$ of the real axis, not passing through the origin, and not intersecting the ray $l:\ \zeta_0 < z < \infty$.

Notation.

1)
$$z = \Phi(w,t), \quad \Phi(\beta(t),t) = 0, \quad \Phi(\infty,t) = \infty, \tag{1}$$
is a function conformally mapping the half-plane $\operatorname{Im} w > 0$ onto the domain $B(t)$, obtained from the plane (z) by making a slit along the curve $\mathcal{L}(t) + l$, where $\mathcal{L}(t) : z = \xi(\tau),\ t_0 \le \tau \le t,\ t < T$, is a part of the curve L.

2) $a_p(t),\ p = 0, 1, 2, \dots, n$, are the preimages of the corner points $z_i,\ i = 0, 1, \dots, \dots, m;\ n = 2m;\ z_0 = \zeta_0$ of the curve $\mathcal{L}(t) + l$, including also the end of the slit, whose preimage we consider to be $a_0(t)$ (to each non-end corner point z_i there correspond two points among the $a_p(t)$) under the mapping (1).

3) $\alpha_p \pi,\ p = 0, 1, 2, \dots, n$, is the angle between the tangents on the left and on the right to the curve $\mathcal{L}(t) + l$ at the points z_i.

4) $M_p(t),\ p = 0, 1, 2, \dots, n$, are the accessory constants of the Schwarz derivative $S(w,t)$ for the function $\Phi(w,t)$ (3).

5) $t_i,\ i = 0, 1, 2, \dots, m$, are the values of the parameter t corresponding to the corner points $z_i,\ z_i = \xi(t_i)$.

Theorem. Let $\chi(w)$ be a function conformally mapping the half-plane $\operatorname{Im} w > 0$ onto the plane with a slit along the ray l. Then the functions

$$\Psi(w,t) = \log \Phi'_w(w,t), \qquad S(w,t) = \Psi''(w,t) - \frac{1}{2}\Psi'^2(w,t)$$

are integrals of the equations

$$\frac{\partial\Psi(w,t)}{\partial t}+\frac{1}{w-a_0(t)}\frac{\partial\Psi(w,t)}{\partial w}=\frac{1}{(w-a_0(t))^2}, \tag{2}$$

$$\frac{\partial S(w,t)}{\partial t}+\frac{1}{w-a_0(t)}\frac{\partial S(w,t)}{\partial w}-\frac{2S(w,t)}{(w-a_0(t))^2}-\frac{6}{(w-a_0(t))^4}=0, \tag{3}$$

satisfying the initial conditions

$$\Psi(w,t_0)=\log\chi'(w), \qquad S(w,t_0)=\Psi''(w,t_0)-\frac{1}{2}\Psi'^2(w,t_0).$$

2. Denote by $\omega(t,t_m)$ the integral of the equation

$$\frac{d\omega(t,t_m)}{dt}=\frac{1}{\omega(t,t_m)-a_0(t)}, \tag{4}$$

which for $t=t_m$ takes the value $a_0(t_m)$. Then equation (4) has:

a) a unique integral $\omega_1(t,t_m)$, $\omega_1(t_m,t_m)=a_0(t_m)$, for which

$$\lim_{t\to t_m+0}\frac{\omega_1(t,t_m)-a_0(t)}{\sqrt{t-t_m}}=\sqrt{\frac{2\alpha_1}{\alpha_n}};$$

b) a unique integral $\omega_2(t,t_m)$, $\omega_2(t_m,t_m)=a_0(t_m)$, for which

$$\lim_{t\to t_m+0}\frac{\omega_2(t,t_m)-a_0(t)}{\sqrt{t-t_m}}=-\sqrt{\frac{2\alpha_n}{\alpha_1}}.$$

Moreover, the relation

$$\lim_{t\to t_m+0}\frac{\omega_1(t,t_m)-a_0(t)}{\omega_2(t,t_m)-a_0(t)}=-\frac{\alpha_1}{\alpha_n}$$

is valid.

3. The curvature $\varkappa(t)$ of the curve $\mathcal{L}(t)+l$ at the point $z=\zeta(t)$ is determined by the formula

$$s'(t)\varkappa(t)=\operatorname{Im}\left\{i\,\frac{a_0(t)-\alpha(t)}{\gamma(t)}+\left(\frac{d\bar\beta(t)}{dt}\right)^2-\frac{d\bar\beta(t)}{dt}\frac{da_0(t)}{dt}+\frac{1}{\pi}\int_{t_0}^{t}\frac{\partial H^*(a_0(t),t,\tau)}{\partial t}\,d\theta(\tau)\right\}, \tag{5}$$

where $s(t)$ is the length of the curve $\mathcal{L}(t)$, measured from the point ζ_0; $\theta(t)$ is the angle formed by the tangent to the curve $\mathcal{L}(t)$ at the point $z=\zeta(t)$ with the real axis; $a(t) = \operatorname{Re}\beta(t)$, $\gamma(t) = \operatorname{Im}\beta(t)$; $\beta(t)$ is the integral of equation (4), assuming at $t = t_0$ the value $\beta(t_0) = \beta_0$, $\operatorname{Im}\beta_0 > 0$, and

$$H^*(a_0(t), t, \tau) = \log \frac{a_0(t) - \omega_2(t,\tau)}{a_0(t) - \omega_1(t,\tau)} \frac{\omega_1(t,\tau) - \beta(t)}{\omega_2(t,\tau) - \beta(t)}.$$

4. What has been said above, as well as what was set forth in (1), makes it possible to propose a method for determining the parameters of the Schwarz derivative $S(w)$ for the function $z = \Phi(w)$, which conformally maps the half-plane $\operatorname{Im} w > 0$ onto a simply connected bounded domain B, whose boundary Γ consists of a finite number of arcs of circles and line segments. It may be assumed, without loss of generality, that the domain B contains the origin and that one of the corner points of the boundary Γ (denote it by $z_0 = \zeta_0$) lies on the positive part of the real axis. We now consider a one-parameter family of domains $B(t)$, obtained from the (z)-plane by making the cut $\mathcal{L}(t) + l$, first along the real axis from $+\infty$ to the point z_0, and then along the curve Γ to the point $z = \zeta(t)$. As $t \to T$, the family $B(t)$ converges to the domain B as to a kernel. Let the corner points z_i, beginning with z_0, be numbered so that a point with a larger index follows a point with a smaller index under positive traversal of the domain B along the boundary Γ. Suppose, further, that the function $z = \Phi(w, t_k)$, conformally mapping the half-plane $\operatorname{Im} w > 0$ onto the domain $B(t_k)$, is known. Then the preimages $a_{p0}^{(k)}$, $p = 0, 1, 2, \dots, 2k-1$, of the corner points of the boundary of the domain $B(t_k)$, and the accessory constants $M_{p0}^{(k)}$, $p = 0, 1, 2, \dots, 2k-1$, of the Schwarz derivative $S(w, t_k)$ are also known. We now extend the cut $\mathcal{L}(t_{k+1}) + l$ to the next corner point of the curve Γ. Then the preimages $a_p^{(k+1)}(t_{k+1})$, $p = 0, 1, 2, \dots, 2k+1$, of the corner points of the curve $\mathcal{L}(t_{k+1}) + l$ under the mapping $z = \Phi(w, t_{k+1})$, and also the accessory constants $M_p^{(k+1)}(t_{k+1})$, $p = 0, 1, 2, \dots, 2k+1$, of the Schwarz derivative $S(w, t_{k+1})$ are determined as integrals of the system (3) from (1), p. 13, satisfying the initial conditions $a_p^{(k+1)}(t_k) = a_{p0}^{(k)}$, $p = 2, 3, \dots, 2k$; $a_j^{(k+1)}(t_k) = a_{00}^{(k)}$, $j = 0, 1, 2k+1$; $M_p^{(k+1)}(t_k) = M_{p0}^{(k)}$, $p = 2, 3, \dots, 2k$;

$$\lim_{t \to t_k + 0} (t - t_k) M_j^{k+1}(t), \qquad j = 0, 1, 2k+1.$$

The system (3) from (1) is integrated by some numerical method (2) from the value of the parameter $t = t_k$ to the value $t = t_{k+1}$. The latter, in turn, is determined from the relation

$$s(t_{k+1}) = \int_{t_0}^{t_{k+1}} \left\{ \frac{1}{4} |\gamma(t)|^{-3} |a_0^{(k+1)}(t) - \beta(t)|^3 \exp\left[-2 \int \left(\frac{d\gamma}{dt}\right)^2 dt\right] \times \right.$$

$$\times \exp \frac{1}{\pi} \int_{t_0}^{t} \operatorname{Re} H^*(a_0^{(k+1)}(t), t, \tau)\, d\theta(\tau) \Bigg\}\, 2 \left(\frac{d\gamma}{dt}\right)^2 dt.$$

For the start of the computation one may use the series

$$a_p^{(k+1)}(t) = a_{p0}^{(k)} + a_{p1}\sqrt{t - t_k} + a_{p2}\left(\sqrt{t - t_k}\right)^2 + \dots, \quad p = 0, 1, 2, \dots, 2k + 1;$$

$$M_p^{(k+1)}(t) = M_{p0}^{(k)} + m_{p1}\sqrt{t - t_k} + m_{p2}\left(\sqrt{t - t_k}\right)^2 + \dots,$$

$$p = 2, 3, \dots, 2k;$$

$$M_j^{(k+1)}(t) = \frac{m_{j,-1}}{\sqrt{t - t_k}} + m_{j0} + m_{j1}\sqrt{t - t_k} + \dots; \quad j = 0, 1, 2k + 1,$$

in which the functions $a_p^{(k+1)}(t)$, $M_p^{(k+1)}(t)$ are expanded. Thus, the constants $a_p^{(k+1)}(t_{k+1})$ and $M_p^{(k+1)}(t_{k+1})$ will be determined with the accuracy allowed by the numerical method. Noting further that the function $z = \Phi(w, t_0)$, mapping the half-plane $\operatorname{Im} w > 0$ onto the plane with a slit along the ray $l : z_0 \le z < \infty$, is known, and that the family of functions $\Phi(w, t)$, as $t \to T$, converges uniformly inside the domain $\operatorname{Im} w > 0$ to $\Phi(w)$, we conclude that, by the method described above, all the parameters entering Schwarz' s equation (3) for the function $z = \Phi(w)$ can be determined.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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