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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

S. S. Sannikov

# ON REPRESENTATIONS OF A CONTINUOUS GROUP BY UNBOUNDED OPERATORS

(Presented by Academician A. I. Mal'cev on 9 XI 1966)

1. In connection with the introduction into quantum theory of complex angular momenta <sup>(1)</sup>, in <sup>(2)</sup> linear representations of the rotation group  $O_3$  were constructed, corresponding to arbitrary values of the angular momentum and spin\*. In contrast to the well-known finite-dimensional representations, to which the classical definition of a representation of a compact continuous Lie group <sup>(4,5)</sup>, based on the concept of a bounded operator, leads, the representations constructed are infinite-dimensional, generally speaking nonunitary and multivalued. Such representations are given by unbounded operators in a certain linear topological space and are connected with a generalization of the concept of a function on a group to a very broad class of singular functions.

In this note elements are given of the theory of representations of a continuous Lie group by unbounded operators, and one example of such a representation is analyzed in the case of the rotation group  $O_3$  (a compact group).

2. Let  $G$  be a Lie group; let  $\mathcal{L}$  be a certain linear topological space on which unbounded operators  $T$  act;  $D_T$  and  $R_T$  are the domains of definition and ranges of the operator  $T$  in  $\mathcal{L}$  ( $D_T$  need not coincide with all of  $\mathcal{L}$ ).

**Definition 1.** A mapping  $g \rightarrow T(g)$ ,  $g \in G$ , where  $T(g)$  are unbounded operators, will be called a **representation of the group  $G$**  in  $\mathcal{L}$ , if  $\mathcal{L} \supset D_{T(g)}$  for every  $T(g)$ , and on the domain of definition the operators  $T(g)$  satisfy all group axioms, in particular

$$T(g_1 g_2) = T(g_1) T(g_2).$$

We shall consider only such mappings for which, for any  $g \in G$ , each set  $D_{T(g)}$  is everywhere dense in  $\mathcal{L}$ , and moreover  $R_{T(g_1)} \cap D_{T(g_2)}$ \*\*\* for any  $g_1, g_2 \in G$ , and for any countable number of sets  $D_{T(g)}$ , the intersection  $\bigcap D_{T(g)}$  is dense in  $\mathcal{L}$ . Here, by a representation is understood, generally speaking, a projective representation

$$T(g_1) T(g_2) = \eta(g_1, g_2) T(g_1 g_2), \quad T(e) = 1; \quad (1)$$

$\eta(g_1, g_2)$  is a numerical function\*\*\*.

**Definition 2.** The elements  $g_1, g_2 \in G$  will be considered **close** if  $T(g_1)f$  is close to  $T(g_2)f$  in the topology of  $\mathcal{L}$  for  $f \in D_{T(g_1)} \cap D_{T(g_2)}$ . Thus a new topology is specified on  $G^{****}$ .

**Definition 3.** The representation  $g \rightarrow T(g)$  is irreducible in  $\mathcal{L}$  if  $\mathcal{L}$  contains no closed subspaces that include the domain of definition of every  $T(g)$ .

\* In <sup>(3)</sup> these representations were used to describe a certain type of unstable state in quantum mechanics.

\*\* Here  $\bigcup_{g \in G} D_{T(g)} = 0$ .

\*\*\* The important question of the existence of a covering of  $G$ , of which the exact mapping is  $T(g)$ , is discussed in a concrete example of a representation of the group  $O_3$  (see below).

\*\*\*\* The introduction of a topology with respect to a given function or class of functions goes back to J. von Neumann <sup>(6)</sup>.

Such an  $\mathcal{L}$  is constructed as follows. Suppose that  $f \in \mathcal{L}$ . Let, for example,  $f$  be one of the elements of the canonical basis of a representation of the Lie algebra of the group  $G$ .\* A dense set in  $\mathcal{L}$  will be the linear span of the orbit  $T(g)f$ . It is formed by finite sums of the form

$$\sum_i A_i T(g_i) f,$$

where  $A_i$  are complex numbers. At the same time certain  $g_0 \in \Delta_f \subset G$  (the set  $\Delta_f$  does not form a subgroup) must be excluded. Indeed,  $T(g)$  are unbounded operators; consequently, there will be such  $g_0$  for which  $f \notin D_{T(g_0)}$ .

Introduce in  $\mathcal{L}$  the weakest locally convex separable topology defined by the family of seminorms  $P_\varphi(f) = |f(\varphi)|$ ,  $f \in \mathcal{L}$ ,  $\varphi \in \mathcal{L}'$  (the  $P_\varphi(f)$  do not take infinite values), where the linear functionals  $f(\varphi)$  have the property that

$$f(T(g)\varphi) = (T^{-1}(g)f)(\varphi),$$

even if  $T(g)\varphi \notin \mathcal{L}'$ . A sequence  $f_n \in \mathcal{L}$  converges to  $f$  if  $P_\varphi(f_n) \rightarrow P_\varphi(f)$  for all  $\varphi \in \mathcal{L}'$ .

**Definition 4.** Two representations  $T_1(g)$  and  $T_2(g)$  in  $\mathcal{L}_1$  and  $\mathcal{L}_2$  shall be called **equivalent** if there exists an operator  $A$ , mapping a dense set in  $\mathcal{L}_1$  onto a dense set in  $\mathcal{L}_2$ , and an operator  $B$ , mapping a dense set in  $\mathcal{L}'_2$  onto a dense set in  $\mathcal{L}'_1$ , such that for elements of the dense set

$$(Af_1)(\varphi_2) = f_1(B\varphi_2).$$

Obviously, all representations defined by unbounded operators are infinite-dimensional.

**3. Example.** As an example, consider one representation of the rotation group  $O_3$  to which the problem of extracting the square root of a spinor leads (7). In the case of spinors of the second rank, extraction of the root leads to the algebra  $A(a_1, a_2)$ , generated by the operators of the (complex) coordinate and momentum of quantum mechanics (8,9):

$$a_1 = z, \quad a_2 = -\frac{d}{dz}, \quad [a_1, a_2] = 1,$$

where  $z$  is a complex variable. In  $A$  there are given linear canonical transformations

$$a_i \rightarrow a'_i = u_i^k a_k$$

with unitary unimodular matrices

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1,$$

forming the group  $SU(2)$ .\*\* For us the following auxiliary result is important.

**Lemma (7).** The transformations  $a_i \rightarrow a'_i$  can be represented as inner automorphisms in  $A$

$$u_i^k a_k = T(u) a_i^{-1} T^{-1}(u), \quad (2)$$

locally isomorphic to the group  $O_3$ .

**Corollary.** Formula (2) defines a projective representation of the group  $O_3$  of the form (1).

Construct a linear representation with operators  $T(u)$  in a certain class of entire analytic functions of the complex variable  $z$ . Let  $f(z) \in D_{T(u)}$ . We define the action of  $T(u)$  on  $f(z)$  by the formula

$$\begin{aligned} T(u)f(z) &= T(u)f(z)T^{-1}(u)(T(u) \cdot 1) = f(T(u)zT^{-1}(u))\chi(z; u) \\ &= f\left(\alpha z - \bar{\beta} \frac{d}{dz}\right)\chi(z; u), \end{aligned} \quad (3)$$

where  $\chi(z; u) = T(u) \cdot 1$  and  $\chi(z; e) = 1$ . The group condition gives an equation for  $\chi(z; u)$

$$T(u_2)\chi(z; u_1) = \chi(z; u_2 u_1), \quad (4)$$

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\* The existence of such representations is guaranteed by the requirements imposed on  $D_{T(g)}$ .

\*\* This consideration also extends to the general case of unimodular transformations with matrices

$$v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

As a result, we arrive at a representation by unbounded operators of the Lorentz group  $\mathcal{L}_4$  (a noncompact group).

where the parameters of the transformation  $T(u_2 u_1)$  are expressed in terms of the parameters of the transformations  $T(u_1)$  and  $T(u_2)$  by the formulas known for the rotation group.

A solution of (4) is

$$\kappa(z; u) = \frac{1}{\sqrt{\alpha}} \exp\left(\frac{1}{2} z^2 \frac{\beta}{\alpha}\right). \quad (5)$$

Consider the class  $\mathcal{L}$  of entire analytic functions of order  $\rho \leq 2$  and type  $0 \leq K < \infty$ , even with respect to the substitution  $z \mapsto -z$ . Introduce in  $\mathcal{L}$  the weakest of the locally convex separable topologies defined by the family of seminorms  $P_\varphi(f) = |[f, \varphi]|$  with respect to the scalar product

$$[f, \varphi] = \int f(z) \overline{(I\varphi(z))} d\mu(z) \quad (6)$$

with measure <sup>(9,10)</sup>

$$d\mu(z) = \pi^{-1} \exp(-z\bar{z}) dz, \quad dz = dx dy. \quad (7)$$

In (6),  $I\varphi(z) = \varphi(iz)$ ,  $f \in \mathcal{L}$ , and  $\varphi \in \mathcal{L}'$ , where  $\mathcal{L}'$  is the class of entire analytic functions of order  $\rho < 2$  and type  $0 \leq K < \infty$ .

Consider the system of functions  $e^{\frac{1}{2}\sigma z^2}$ , where  $\sigma$  is a complex variable. These functions form a complete system in  $\mathcal{L}$ . Indeed, any function  $f(z) \in \mathcal{L}$  can be represented in the form

$$f(z) = \int e^{\frac{1}{2}\sigma z^2} \tilde{f}(2\sigma) d\mu(\sigma), \quad (8)$$

where the integration is over the whole complex plane ( $\sigma$ ) with measure (7); moreover, we have  $\tilde{f}(z^2) = f(z)$ . The inverse of formula (8) has the form (choosing the sheet on which  $(\sqrt{\sigma})^2 = \sigma$ )

$$\tilde{f}(2\sigma) = \int \text{ch}(\bar{z}\sqrt{2\sigma}) f(z) d\mu(z). \quad (9)$$

Since

$$\int \operatorname{ch}(\bar{z}\sqrt{2\sigma})e^{\frac{1}{2}\bar{\sigma}w^2} d\mu(\sigma) = \operatorname{ch}(\bar{z}w), \quad \int e^{\frac{1}{2}\bar{\sigma}z^2} \operatorname{ch}(\bar{z}\sqrt{2\tau}) d\mu(z) = e^{\bar{\sigma}\tau}, \quad (10)$$

and for any entire analytic functions  $f(z) = f(-z)$  and  $\tilde{f}(\sigma)$  we have

$$\int \operatorname{ch}(z\bar{w})f(w) d\mu(w) = f(z), \quad \int e^{\sigma\bar{\tau}}\tilde{f}(\tau) d\mu(\tau) = \tilde{f}(\sigma),$$

this proves the completeness of the system of functions  $e^{\frac{1}{2}\sigma z^2}$  in  $\mathcal{L}$ . As follows from (9), (10), the systems of functions  $e^{\frac{1}{2}z^2\bar{\sigma}}$ ,  $\operatorname{ch}(\sqrt{2\sigma}\bar{z})$  have biorthogonality properties.

Next we have

$$T(u)e^{\frac{1}{2}z^2\sigma} = \eta(u, \sigma)e^{\frac{1}{2}z^2\sigma(u)}, \quad (11)$$

i.e., under the action of  $T(u)$  the functions  $e^{\frac{1}{2}z^2\sigma}$  pass into expressions of the same form. Computations give

$$\eta(u, \sigma) = \frac{1}{\sqrt{-\beta\sigma + \alpha}}, \quad \sigma(u) = \frac{\alpha\sigma + \beta}{-\beta\sigma + \alpha}, \quad \sigma \equiv \sigma(e). \quad (12)$$

By the definition of the class  $\mathcal{L}$ , values  $\sigma(u) = \infty$  must be excluded. Therefore, for fixed  $u$ , the domain of definition of the operator  $T(u)$  is not the whole system of functions complete in  $\mathcal{L}$ , but only those functions of the form (11) for which  $\sigma \neq \bar{\alpha}/\bar{\beta}$ . Consequently,  $T(u)$  are unbounded operators.\*

\*  $\mathcal{L}$  can be completed by adjoining to it “ideal” elements to which  $\sigma = \infty$  corresponds. Formulas (11), (12) show how to extend  $T(u)$  to such elements. As a result we obtain a space  $[\mathcal{L}]$  invariant with respect to the representation  $T(u)$ .

It is not difficult to verify that  $\mathcal{L}$  contains the domain of definition of each  $T(u)$  and contains no subspaces possessing the same property. Thus, the following holds.

**Theorem.** *The mapping  $u \rightarrow T(u)$  (3), as a linear irreducible representation of the group  $SU(2)$ , is realized by unbounded operators in the class of entire analytic functions of order  $\rho \leq 2$  and type  $0 < K < \infty$ .*

We shall call the following functions on the group the **matrix elements of the representation**:

$$D_{\tau, \bar{\sigma}}(u) = \int \text{ch}(\sqrt{2\tau\bar{z}}) (T(u)e^{\frac{1}{2}z\bar{\sigma}}) d\mu(z) = \eta(u, \bar{\sigma})e^{\bar{\tau}\sigma(u)}. \quad (13)$$

These are singular functions on the group. As formulas (11), (12), (13) show, the representation under consideration is two-valued on the group  $SU(2)$  and four-valued on the group  $O_3$ . Indeed, in the Euler parametrization

$$\alpha = \cos \frac{\vartheta}{2} e^{-i(\varphi_1 + \varphi_2)/2}, \quad \beta = -i \sin \frac{\vartheta}{2} e^{i(\varphi_1 - \varphi_2)/2}.$$

The functions (13) are single-valued on the covering space in which the Euler angles  $\varphi_1, \varphi_2, \vartheta$  take values in the region  $0 \leq \varphi_1, \varphi_2 < 8\pi, 0 \leq \vartheta \leq \pi$ . The group space of such a covering is a certain three-dimensional manifold in an infinite-dimensional space.

In conclusion, we note that with respect to a complete system in  $\mathcal{L}$  and to the matrix elements, the approach considered here differs from the canonical one <sup>(2)</sup>, which makes substantial use of infinitesimal methods.

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*Note: Figure translations are in progress. See original paper for figures.*

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