ON THE THEORY OF DISTAL MINIMAL SETS AND DISTAL FUNCTIONS

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Abstract

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ON THE THEORY OF DISTAL MINIMAL SETS AND DISTAL FUNCTIONS

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1°. Let (X, ρ) be a compact metric space; J the group of real numbers; (X, J, π) a dynamical system ([1]) (a group of transformations ([2])). The system (X, J, π) is called **distal** ([3]) if, for any two distinct points x and y of X, there is a number $d > 0$ such that $\rho(xt, yt) > d$ for all $t \in J$. Every equicontinuous ([2]) (uniformly Lyapunov stable ([1])) system is, obviously, distal.

In the work of Auslander, Hahn, and Markus ([4]) it was shown that on every nilmanifold ([5]) X one can define a dynamical system (X, J, π) under which X is a distal minimal set, but, generally speaking, is not equicontinuous.

In the present note it is proved that on every torus T^α (finite-dimensional or infinite-dimensional) one can define a distal dynamical system under which T^α is a minimal set. An example is given of a distal non-equicontinuous system on the three-dimensional torus T^3, under which T^3 is a minimal set. Further, it is proved that if $\varphi(t)$ and $\psi(t)$ $(t \in J)$ are Bohr almost-periodic functions ([6]), then the function

$$\varphi\left(\int_0^t \psi(\tau)\, d\tau\right)$$

is distal ([7,8]), but, generally speaking, is not Bohr almost-periodic.

2°. In what follows, by T we shall denote the factor group of the group J by the subgroup generated by the number 2π.

Theorem 1. *A dynamical system defined on the n-dimensional torus T^n by the system of differential equations*

$$dx_1/dt = \varphi_1,$$

$$dx_i/dt = \varphi_i(x_1, \dots, x_{i-1}) \quad (i = 2, \dots, n), \tag{1}$$

where $\varphi_i \in J$, $\varphi_i(x_1, \dots, x_{i-1})$ are functions periodic with period 2π in each variable and is distal. For the system (1) there exist numbers $\gamma_2, \dots, \gamma_n$ such that the system of equations

$$dx_1/dt = \varphi_1,$$

$$dx_i/dt = \varphi_i(x_1, \dots, x_{i-1}) + \gamma_i \quad (i = 2, \dots, n) \tag{1'}$$

defines a dynamical system under which T^n is a minimal set.

Proof. The proof of Theorem 1 is carried out by induction on n. For $n = 1$ the assertion is obvious. Suppose the assertion is true for $n = k$, and let $a = (x_1, \dots, x_{k+1})$ and $b = (y_1, \dots, y_{k+1})$ be two distinct points of the torus T^{k+1}. If there is an index i such that $1 \le i \le k$ and $x_i \ne y_i \pmod{2\pi}$, then the points a and b are distal by virtue of the induction hypothesis. If, however, $x_i = y_i \pmod{2\pi}$ for all i, $1 \le i \le k$, then $x_{k+1} \ne y_{k+1} \pmod{2\pi}$, and in this case the distance between the points at and bt does not depend on $t \in J$.

By virtue of the induction assumption we may suppose that the system of equations (1) defines on the k-dimensional torus $T^k = \{(x_1, \dots, x_k)\}$ a dynamical system for which T^k is a distal minimal set. It is easy to prove that the intersection of the closure of the trajectory of the point $p = (0, \dots, 0) \in T^{k+1}$ with the circle $(0, \dots, 0, x_{k+1})$ is a subgroup of this group. To complete the proof of the second assertion of the theorem it suffices to use the following proposition.

Lemma. Let X be a compactum and let p be a homomorphic mapping of a distal dynamical system (X, J, π) onto a system (Y, J, ρ), in which Y is a minimal set. If there exists a point $x \in X$ such that the closure of the trajectory of the point x contains the set $p^{-1}(p(x))$, then X is a minimal set for (X, J, π).

Theorem 1 can obviously be generalized also to infinite-dimensional tori.

We now show that on any torus T^n $(n \geqslant 3)$ one can define a dynamical system for which T^n is a distal, not uniformly continuous, minimal set. By virtue of Theorem 1 it suffices to construct an example of such a system on T^3.

Example. In ([1]), on p. 424, it is shown that there exist an irrational number μ and a function $\Phi(x_1, x_2)$, continuous on T^2, such that for any $m > 0$ and $\varepsilon > 0$ one can specify $a \in T$ and $t \in J$ for which

$$\left| \int_0^t \Phi(\tau, \mu\tau)\, d\tau - \int_0^t \Phi(\tau, a + \mu\tau)\, d\tau \right| > m, \qquad |a| < \varepsilon \pmod{2\pi}. \tag{2}$$

By virtue of Theorem 1 there exists a number γ such that the system of equations

$$\begin{aligned} &dx_1/dt = 1, \\ &dx_2/dt = \mu, \\ &dx_3/dt = \Phi(x_1, x_2) + \gamma \end{aligned} \tag{3}$$

defines on T^3 a distal dynamical system for which T^3 is a minimal set. From condition (2) it follows that the system (3) is not uniformly continuous.

From the results of the paper ([9]) the following assertion follows.

Theorem 2. If the torus T^2 is a distal minimal set for the system (T^2, J, π), then this system is uniformly continuous.

However, the torus T^2 may be a distal, not uniformly continuous, minimal set for a discrete system.

3°. A continuous real function $\varphi(t)$ $(t \in J)$ is called **distal** ([7,8]) if the closure of the trajectory of the point $\varphi(t)$ in the Bebutov system ([10]) is a compact distal minimal set. It is known ([7]) that the set of all distal functions forms an algebra. It is easy to prove that the limit of a uniformly convergent sequence of distal functions is a distal function. From ([11]) it follows that, for any distal function $\Phi(t)$, the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \Phi(t+a)\, dt$$

exists uniformly in $a \in J$.

Denote by B the set of all Bohr almost-periodic functions.

Theorem 3. If $\psi \in B$, then the set

$$B_\psi = \left\{ \varphi \left(\int_0^t \psi(\tau)\, d\tau \right) \mid \varphi \in B \right\}$$

forms an algebra of distal functions.

Proof. It is known ([6]) that for every function $\psi \in B$ there exist a function $F(x_1, \dots, x_n, \dots)$, continuous on T^∞, and numbers $\lambda_1, \dots, \lambda_n, \dots$ such that

$$\psi(t) = F(\lambda_1 t, \dots, \lambda_n t, \dots).$$

The system

$$\begin{aligned} dx_i/dt &= \lambda_i \qquad (i = 1, \dots, n, \dots), \\ dy/dt &= F(x_1, \dots, x_n, \dots) \end{aligned}$$

is distal. Therefore, for any function $\varphi(y)$ periodic with period 2π, the function

$$\varphi \left(\int_0^t F(\lambda_1 \tau, \dots, \lambda_n \tau, \dots)\, d\tau \right) = \varphi \left(\int_0^t \psi(\tau)\, d\tau \right)$$

is distal. To complete the proof it suffices to use the remarks made at the beginning of this section.

The example constructed above shows that functions of the form

$$\varphi\left(\int_0^t \psi(\tau)\,d\tau\right),$$

($\varphi, \psi \in B$), generally speaking, are not almost periodic in the sense of Bohr. Moreover, they are not, generally speaking, almost periodic in the sense of Levitan ([6]), as the following assertion shows.

Theorem 4. *If a function almost periodic in the sense of Levitan is distal, then it is almost periodic in the sense of Bohr.*

In conclusion we note that, using Theorem 1, one can also indicate other classes of distal functions.

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Note: Figure translations are in progress. See original paper for figures.