Nonlinear differential equations in a Banach space, close to linear ones

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Abstract

The article investigates the exponential growth rates of solutions to the equations

$$\frac{dx}{dt} = Ax, \tag{1}$$

$$\frac{dx}{dt} = Ax + h(x,t), \tag{2}$$

where A is a linear unbounded operator in a complex Banach space X, and $h(x,t)$ is a nonlinearity. It is assumed that the operator A is the generator of a semigroup e^{At} that is strongly continuous for $t \geq 0$.

If $S(A)$ is the set of finite growth rates of solutions to (1), and e^{At} possesses the property $e^{A\xi_0}X \subset D(A)$ for some $\xi_0 > 0$, and the resolvent $R(\lambda; A)$ grows no faster than a power, then the following result holds: $S(A) \subset \operatorname{Re}\sigma(A)$, and if ξ is an isolated point of $\operatorname{Re}\sigma(A)$, then $\xi \in S(A)$. Here $\operatorname{Re}\sigma(A)$ is the set of real parts of the points in the spectrum of A. More precise results are obtained by assuming A to be a scalar-type operator.

The connection between the growth rates of solutions to (1) and (2) and the conditions for their proximity are studied under the assumption that the operator A is the generator of an analytic semigroup, and $h(x,t)$ is subject to certain conditions. The main result of this part of the work is given by the following theorem: For any $\varepsilon > 0$ and $T > 0$, there exists $\delta = \delta(\varepsilon, T) > 0$ such that if

$$\int_{t_0}^{t} \frac{e^{\varepsilon(\tau - t)}\gamma(\tau)}{(t-\tau)^\alpha}\, d\tau + \int_{t}^{\infty} e^{\varepsilon(t-\tau)}\gamma(\tau)\, d\tau < \delta, \quad t > t_0,$$

then the growth rate of any solution to (2) either does not exceed $b - T$, where $b = \sup \operatorname{Re}\sigma(A)$, or is located at a distance from $\operatorname{Re}\sigma(A)$ of no more than ε.

Bibliography: 12 items.

Full Text

Preamble

This work, following the developments in [11] and [12], investigates the differential equation

$$H = Ax + h(x,t)$$

where A is a linear operator and $h(x,t)$ is a nonlinear term. We consider the case where A is a closed operator with a domain $D(A)$ dense in a Banach space X. In Section 1, we establish the fundamental properties of the operator A and the associated semigroup e^{At}. Specifically, we analyze the relationship between the spectrum $\sigma(A)$ and the growth exponent of the solution to the linear equation

$$\dot{x} = Ax, \quad x(0) = x_0. \tag{0.1}$$

The characteristic exponent $\chi(x)$ for a solution $x(t)$ is defined as

$$\chi[x] = \limsup_{t\to\infty} \frac{\ln \|x(t)\|}{t}.$$

We denote the set of all such exponents as $S(A) = \{\chi(x) : x \in X, x \neq 0\}$. It is well known that for a bounded operator A, the supremum of $S(A)$ coincides with the spectral radius, specifically $\sigma(A) = \sup \operatorname{Re}\, \sigma(A)$. However, for unbounded operators, this relationship is more complex. As shown in [6] and [10], the upper bound of the spectrum $\sigma(A)$ is defined as:

$$\sigma(A) = \limsup_{t\to\infty} \frac{\ln \|e^{At}\|}{t} \tag{1.1}$$

where $\sigma(A) = \sup S(A) \geq \sup \operatorname{Re}\, \sigma(A)$.

Section 1. Spectral Properties and Semigroups

We consider the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$. For an operator A generating a strongly continuous semigroup, the growth of the semigroup is constrained by the resolvent’ s behavior in the complex plane. If A satisfies the conditions for an analytic semigroup, then for any $\epsilon > 0$, there exists a constant M_ϵ such that:

$$\|e^{At}x\| \leq M_\epsilon e^{(\sigma(A)+\epsilon)t} \|x\| \tag{1.2}$$

for all $t > 0$ and $x \in X$.

In cases where the space X can be decomposed into invariant subspaces $X = X_1 \oplus X_2$ corresponding to different parts of the spectrum $\sigma(A) = \sigma_1 \cup \sigma_2$, the operator A can be represented as a direct sum $A = A_1 \oplus A_2$. If $\operatorname{Re}\, \sigma_1 < \xi < \operatorname{Re}\, \sigma_2$, then the solutions $x(t)$ can be partitioned accordingly, allowing for precise asymptotic estimates of the components in X_1 and X_2.

For $x \in D(A^{m+2})$, the solution to the inhomogeneous problem can be represented using the Dunford integral:

$$e^{At}x = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} R(\lambda; A)x \, d\lambda \tag{1.3}$$

where γ is a suitable contour in the resolvent set. As demonstrated in [10], if the resolvent satisfies certain growth conditions, specifically $\|R(\lambda; A)\| \leq K(1 + |\lambda|)^m$, the integral representation converges for $t > 0$. This allows us to define the fractional powers of the operator A^α and establish estimates of the form:

$$\|A^\alpha e^{At}x\| \leq Mt^{-\alpha} e^{\sigma(A)t} \|x\| \tag{1.5}$$

Section 2. Nonlinear Perturbations and Stability

We now turn to the nonlinear equation:

$$\dot{x} = Ax + h(x, t) \tag{0.2}$$

where $h(x, t)$ satisfies a Lipschitz condition $\|h(x_1, t) - h(x_2, t)\| \leq L\|x_1 - x_2\|$. We assume that $h(0, t) = 0$, ensuring that $x = 0$ is a trivial solution. Using the variation of constants formula, the solution can be written in the integral form:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} h(x(\tau), \tau) \, d\tau \tag{2.9}$$

Applying the estimates from Section 1, we can prove the existence and uniqueness of solutions in the space $D(A^\alpha)$. Specifically, if $x_0 \in D(A^\alpha)$, then for a sufficiently small time interval $[t_0, t_0 + T]$, there exists a unique solution $x(t)$ such that:

$$\|A^\alpha x(t)\| \leq K(t - t_0)^{-\alpha} \|x_0\| + \int_{t_0}^{t} (t - \tau)^{-\alpha} \|h(x(\tau), \tau)\| \, d\tau$$

By applying Gronwall’ s inequality, we establish the stability of the trivial solution under the condition that the linear part A is exponentially stable, i.e., $\sigma(A) < 0$.

Section 3. Asymptotic Behavior

In this section, we analyze the characteristic exponents of the nonlinear system (0.2). Let $\sigma(A)$ be the spectral bound of the linear operator. We show that if the nonlinear term $h(x, t)$ is small in a certain sense, specifically if:

$$\|h(x, t)\| \leq \gamma(t) \|A^\alpha x\| \tag{3.1}$$

where $\gamma(t) \to 0$ as $t \to \infty$, then the characteristic exponents of the nonlinear system are governed by the spectrum of A.

Theorem 3.1. For any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x_0\| < \delta$, the solution $x(t)$ to (0.2) satisfies:

$$\chi[x] \leq \sigma(A) + \epsilon$$

This result implies that the stability of the nonlinear system is robust to perturbations that satisfy the growth conditions defined in (3.1). The proof utilizes a contraction mapping argument in the space of continuous functions with exponential weight, combined with the fractional power estimates derived in Section 2.

Furthermore, we consider the case where the spectrum $\sigma(A)$ is divided by a gap. If the nonlinear perturbation is sufficiently small, the manifold structure of the linear system persists. Specifically, there exist invariant manifolds in X that are tangent to the spectral subspaces X_1 and X_2 at the origin. This allows for a local decoupling of the dynamics, facilitating the study of conditional stability and the existence of bounded solutions on the half-line $[t_0, \infty)$.

Note: Figure translations are in progress. See original paper for figures.