COMPARISON THEOREMS FOR FRACTIONAL POWERS OF OPERATORS

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Abstract

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COMPARISON THEOREMS FOR FRACTIONAL POWERS OF OPERATORS

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1. In this paper a perturbation theory is developed for fractional powers of weakly positive operators acting in a Banach space E (w.p. E-operators) *. The results obtained make it possible to establish how the fractional powers of elliptic operators change when the coefficients of the equations and the boundary conditions are changed.

2. Let A_1 and A_2 be w.p. E-operators. Let

$$\Phi(v,w) \equiv (A_1 v_1, w) - (v, A_2^* w) = \sum_{i=1}^{m} \Phi_i(v,w) \qquad (v \in D(A_1),\ w \in D(A_2^*)). \tag{1}$$

Here (φ, ψ) is the value of the functional ψ from the conjugate E-space E^*; A_2^* is the operator acting in E^* conjugate to A_2; $\Phi_i(v,w)$ are bilinear forms defined on $D(A_1) \times D(A_2^*)$, satisfying, for certain δ_i and ρ_i in $[0,1]$, the inequalities

$$|\Phi_i(v,w)| \le R_i \|A_1 v\|^{\delta_i} \|v\|^{1-\delta_i} \|A_2^* w\|^{\rho_i} \|w\|^{1-\rho_i}. \tag{2}$$

Let $-\min \delta_i < s < 1 - \max \delta_i$, if all $\delta_i \in (0,1)$. If, however, $\min \delta_i = 0$ or $\max \delta_i = 1$, then in the corresponding place the sign $<$ must be replaced by the sign $\le$. Let τ satisfy the same conditions with respect to the system of numbers ρ_i.

Theorem 1. Let $-1 < \alpha < 1 - \tau - \max \rho_i$. Let the numbers $\Delta_i = \rho_i + \delta_i + \tau + s + \alpha$ be renumbered so that $\Delta_i > 1$ for $i = 1, \dots, l$, $\Delta_i = 1$ for $i = l+1, \dots, k$, and $\Delta_i < 1$ for $i = k+1, \dots, m$. Then for any $0 < \varepsilon_i < 1 - \delta_i - s$, $i = l+1, \dots, k$, and $v \in D(A^\gamma)$, $\gamma = \max(1, \alpha + s)$, the inequality

$$\|A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s v\| \le c \left[\sum_{i=1}^{l} R_i (\Delta_i - 1)^{-1} (1 - \rho_i - \tau - \alpha)^{-1} \|A_1 v\|^{\Delta_i - 1} \times \right.$$

$$\times\|v\|^{2-\Delta_i} + \sum_{i=l+1}^{k} R_i\varepsilon_i^{-1}\|A_i^{\varepsilon_i}v\| + \sum_{i=k+1}^{m} R_i(1-\Delta_i)^{-1}\|v\|\Bigg]. \tag{3}$$

Moreover, if $\tau < 0$ and $\alpha > 0$, then it is understood that

$$A_2^\tau(A_1^\alpha - A_2^\alpha)A_1^s = A_2^\tau A_1^{\alpha+s} - A_2^{\tau+\alpha}A_1^s.$$

Let $\max\rho_i < 1$ and $\theta = \max(\rho_i + \delta_i)$. Then it follows from (3) that, for any $\max(0, 1-\theta) < \alpha < 1 - \max\rho_i$, $\varepsilon > 0$, the inequality

$$\|(A_1^\alpha - A_2^\alpha)v\| \le cR\varepsilon^{-1}\|A_1^{\theta+\alpha+\varepsilon-1}v\| \qquad (R = \max R_i) \tag{4}$$

holds.

* For the basic results on w.p. E-operators see ([1–6]); some of their proofs see in ([7]). Perturbation theories in Hilbert space are devoted to ([8–10]).

Let $\theta \geqslant 1$; then it follows from this that $D(A_1^{\theta+\alpha+\varepsilon-1}) \subset D(A_2^\alpha)$ for every $\varepsilon > 0$. Let $\theta < 1$; then from (4) it follows that $D(A_1^\alpha) \subset D(A_2^\alpha)$ and the operator $A_1^\alpha - A_2^\alpha$ is subordinate to the operator $A_1^{\alpha-\delta}$, where $\delta = 1 - \theta - \varepsilon > 0$, since $\varepsilon > 0$ is arbitrary. Finally, if $\theta < 1$ and $0 \leqslant \alpha < 1 - \theta$, then

$$\|(A_1^\alpha - A_2^\alpha)v\| \leqslant cR(1-\theta-\alpha)^{-1}\|v\|. \tag{5}$$

i.e., the operator $A_1^\alpha - A_2^\alpha$ admits closure to a bounded operator.

Let us consider the general example. Let A_1 and A_2 be s.p. E-operators, $D(A_1) = D(A_2)$, and suppose that for some $0 < \delta < 1$ the inequality

$$\|(A_1 - A_2)v\| \leqslant R\|A_iv\|^\delta\|v\|^{1-\delta} \qquad (v \in D(A_i), \quad i = 1 \text{ or } i = 2) \tag{6}$$

holds.

Then $D(A_1^\alpha) = D(A_2^\alpha)$ for $0 \leqslant \alpha \leqslant 1$, and the inequalities

$$\|(A_1^\alpha - A_2^\alpha)v\| \leqslant cR\varepsilon^{-1}\|A_i^{\alpha+\delta+\varepsilon-1}v\| \qquad (1-\delta \leqslant \alpha \leqslant 1,\ \varepsilon > 0),$$

$$\|(A_1^\alpha - A_2^\alpha)v\| \leqslant cR(1-\delta-\alpha)^{-1}\|v\| \qquad (0 \leqslant \alpha < 1-\delta). \tag{7}$$

are valid.

3. The results of item 2 can be refined in the case when A_1 and A_2 are positive definite self-adjoint operators.

Suppose that (1) is fulfilled ($A_2^* = A_2$) and that $\Phi_i(v, w)$ satisfy the stronger condition than (2),

$$|\Phi_i(v, w)| \leqslant R_i \|A_1^{\delta_i} v\| \cdot \|A_2^{\rho_i} w\|. \tag{8}$$

Theorem 2. Suppose

$$-\min \delta_i \leqslant s \leqslant 1 - \max \delta_i, \qquad -\min \rho_i < \tau < 1 - \max \rho_i,$$

$$-\min \rho_i - \tau < \alpha < 1 - \tau - \max \rho_i.$$

Then, for any $v \in D(A^\gamma)$, $\gamma = \max(1, \alpha + s)$, the inequality

$$\|A_2^\tau (A_1^\alpha - A_2^\alpha) A^s v\| \leqslant c |\sin \alpha| \sum_{i=1}^m [(1 - \max \rho_i - \tau)(\tau + \min \rho_i)$$

$$\times (1 - \tau - \max \rho_i - \alpha)(\alpha + \tau + \min \rho_i)]^{1/2} \|A_1^{\Delta_i - 1} v\|. \tag{9}$$

Here, if $\tau < 0$ and $\alpha > 0$, then it is assumed that

$$A_2^\tau (A_1^\alpha - A_2^\alpha) A_1^s \equiv A_2^\tau A_1^{\alpha+s} - A_2^{\tau+\alpha} A_1^s.$$

In particular, it follows from this that for $0 < \min \rho_i$, $\max \rho_i < 1$, $0 \leqslant \alpha < 1 - \max \rho_i$, and $\theta = \max(\rho_i + \delta_i)$, the inequality

$$\|(A_1^\alpha - A_2^\alpha) v\| \leqslant cR \|A_1^{\alpha+\theta-1} v\| \qquad (R = \max R_i). \tag{10}$$

is valid.

It follows that $D(A_1^\alpha) \subset D(A_2^\alpha)$ for $\theta \leqslant 1$ and $0 \leqslant \alpha < 1 - \max \rho_i$. If $\theta < 1$, then the operator $A_1^\alpha - A_2^\alpha$ is subordinate to the operator $A_1^{\alpha-\delta}$, where $\delta = 1 - \theta > 0$. Therefore, for all $0 \leqslant \alpha \leqslant 1 - \theta$, the operator $A_1^\alpha - A_2^\alpha$ admits closure to a bounded operator.

Let us consider the general example. Let A_1 and A_2 be positive definite self-adjoint operators, $D(A_1) = D(A_2)$, and suppose that for some $0 \leqslant \delta \leqslant 1$ the inequality

$$\|(A_1 - A_2) v\| \leqslant R \|A_i^\delta v\|. \tag{11}$$

holds.

Then (cf. (8)) from Theorem 1 and (11) it follows that for any $0 \leqslant \alpha \leqslant 1$

$$\|(A_1^\alpha - A_2^\alpha) v\| \leqslant cR \|A_i^{\alpha+\delta-1} v\|. \tag{12}$$

Remark 1. It follows from Theorem 1 that all assertions of this item are valid for s.p. H-operators A_i, differing from self-adjoint ones by subordinate operators.

4. Let $A(t)$ be an operator-valued function defined on $[0, T]$ with values in the set of s.p. E of operators. Theorems 1 and 2 make it possible to estimate the smoothness of the operator-valued function $A^\alpha(t)$ in terms of the smoothness of the operator-valued function $A(t)$. We give only a theorem on the differentiability of $A^\alpha(t)$.

Theorem 3. Let the operator-valued function $[A(t) + \lambda I]^{-1}$ be strongly differentiable with respect to t on $[0, T]$, and let the form $\Phi(v, w) = -(A[t_1]v, w) - (v, A^*[t_2]w)$ and the numbers s and τ satisfy the conditions of item 2 for $m = 1$ and $R_1 = c|t_1 - t_2|$. Then, for every $-1 < \alpha < 1 - \rho_1$, the operator-valued function $A^\alpha(t)A^{-1}(0)$ is strongly differentiable with respect to t on $[0, T]$, and the inequalities

$$\|A^\tau(0)[A^\alpha(t)]' A^s(0)v\| \le \begin{cases} c(\Delta_1 - 1)^{-1}(1 - \rho_1 - \tau - \alpha)^{-1}\|A(0)v\|^{\Delta_1 - 1}\|v\|^{2-\Delta_1} & (1 < \Delta_1 < 2), \\ c\varepsilon^{-1}\|A^\varepsilon(0)v\| & (\Delta_1 = 1,\ \varepsilon > 0), \\ c(1 - \Delta_1)^{-1}\|v\| & (\Delta_1 < 1). \end{cases} \tag{13}$$

hold.

If, however, $A(t)$ for each $t \in [0, T]$ is a positive definite self-adjoint operator, and the form $\Phi(v, w)$ and the numbers s and τ satisfy the conditions of item 3 for $m = 1$ and $R_1 = C|t_1 - t_2|$, then the inequality

$$\|A^\tau(0)[A^\alpha(t)]' A^s(0)v\| \le c|\sin \pi\alpha| \times \tag{14}$$

$$\times \left[(1 - \rho_1 - \tau)(\tau + \rho_1)(1 - \tau - \rho_1 - \alpha)(\alpha + \tau + \rho_1)\right]^{1/2} \left\|A^{\Delta_1 - 1}(0)v\right\|$$

is valid.

Remark 2. The last assertion of the theorem is valid for s.p. H of operators $A(t)$ differing from a self-adjoint operator by a subordinate operator.

Remark 3. If the form $\Phi(v, w)$ satisfies the conditions of item 2 for $m = 1$, $\rho_1 + \delta_1 \le 1$, and $R_1 \to 0$ as $t_1 - t_2 \to 0$, then for all $t \in [0, T]$ the inequality

$$\left\|[A(t) + \lambda I]^{-1}\right\| \le C(1 + \lambda)^{-1}$$

is valid.

5. Let Ω be a domain of n-dimensional space with boundary S. Consider an operator A acting in $L_p(\Omega)$ $(1 < p < \infty)$, defined by the elliptic differential expression

$$-\sum_{i,k=1}^{n} \left[a_i^k(x) v'_{x_k}\right]'_{x_i} + \sum_{i=1}^{n} a_i(x) v'_{x_i} + a(x)v \quad (x \in \Omega) \tag{15}$$

on functions from $W^{(2)}_{p,A}(\Omega)$ satisfying the boundary condition

$$-\sum_{i,k=1}^{n} a_{ik}(x)v'_{x_k}\cos(N,x_i)+\sigma(x)v=0 \quad (x\in S). \tag{16}$$

By B we shall denote the operator generated by the boundary condition $v=0$ $(x\in S)$. If $a(x)\geq a_0$ and a_0 is sufficiently large, then A and B are s.p. $L_p(\Omega)$-operators (see, for example, (12)). If the coefficients of (15) and (16) depend, respectively, on t and z, then we obtain operators $A(t,z)$ and $B(t)$.

Theorem 4. If $0\leq\alpha<1-1/2q$ $(1/q+1/p=1)$, then

$$A^{\alpha}(t,z_1)-A^{\alpha}(t,z_2)$$

is subordinate to the operator

$$A^{\alpha-1/2+\varepsilon}(t,z_3)$$

$(\varepsilon>0)$. If $p=2$, then $\varepsilon=0$ and $D[A^{\alpha}(t_1,z)]=D[B^{\alpha}(t_2)]$ for $0\leq\alpha<1/4$.

The proof uses the results from $(13,14)$. The equality

$$D[A^{\alpha}(t_i,z)]=D[B^{\alpha}(t_2)]$$

also follows from (15).

Theorem 3 makes it possible to prove the differentiability of $A^{\alpha}(t,z)A^{-1}(t_0,z_0)$ and $B^{\alpha}(t)B^{-1}(t_0)$, if the coefficients of (15) and (16) are differentiable with respect to t and z.

6. Consider semibounded elliptic operators A_1 and A_2 of order $2m$ with normal boundary conditions. If B is a differential operator of order $k<2m$, then the operator $(A_i+B)^{\alpha}-A_i^{\alpha}$ is subordinate in $L_p(\Omega)$ to the operator $A_i^{\alpha+k/2m-1+\varepsilon}$ $(\varepsilon>0)$, where $\varepsilon=0$ in the case of the self-adjoint operator A_i in $L_2(\Omega)$. If A_i are generated by the same

by a differential expression and boundary conditions with identical principal parts, then there exist such μ,ν in $(0,1)$ that, for $0\leq\alpha<\mu$, the operator $A_1^{\alpha}-A_2^{\alpha}$ is subordinate to the operator $A_i^{\alpha-\nu+\varepsilon}$ $(\varepsilon>0)$.

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Note: Figure translations are in progress. See original paper for figures.