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On the problem of controllability of linear systems with aftereffect

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Abstract

Full Text

Preamble

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This paper considers the problem of controlling dynamical systems with a lagging argument. We introduce the concept of a relatively controllable system, which is necessitated by the specific nature of differential equations with delays. The questions of relative controllability for linear systems are investigated, and conditions ensuring the controllability of certain linear systems with delay are provided. The results obtained are closely related to the theory of optimal processes [**?**].

1. Basic Concepts

Let Z be the state space of a dynamical system; U be the set of control actions (controls); and $z = z(z_0, u, t)$ be the vector characterizing the state of the dynamical system at time t, determined by the initial condition $z_0 \in Z$ at $t = t_0$ and the control $u \in U$. We denote a subspace of Z by Z_1, and the projection of the state vector $z(z_0, u, t)$ onto Z_1 by $pr_{Z_1} z(z_0, u, t)$.

Definitions

1. A state z_0 is called **controllable** in class U (a **controllable state**) if there exist a control $u = u(z_0, U, t)$ and a time T, $t_0 < T < \infty$, such that $z(z_0, u, T) = 0$.
2. A state z_0 is called **controllable** in class U **relative to a given set** Z_1 (a **relatively controllable state**) if there exist a control $u \in U$ and a time T, $t_0 < T < \infty$, such that $pr_{Z_1} z(z_0, u, T) = 0$.
3. If every state $z_0 \in Z$ of the dynamical system is controllable, we say that the system is **controllable** (a **controllable system**). Similarly, a

relatively controllable system is understood as a dynamical system in which every state is relatively controllable.

Consider the system:

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t-h) + Cu(t), \quad t \in [0,T], \quad T < \infty \quad (1.1)$$

with initial conditions $x(0) = x_0$, $x(\theta) = \phi(\theta)$, $-h \leq \theta < 0$, where $x = \{x_1, \dots, x_n\}$ is the vector of phase coordinates, $x \in X_n$; $u(t) = [u_1(t), \dots, u_r(t)]$ is the control, $u \in U$, where U is the set of piecewise continuous functions; A, B, C are constant matrices of dimensions $(n \times n)$, $(n \times n)$, and $(n \times r)$ respectively; and h is a constant delay. The state space Z of the system under consideration is the set of n-dimensional vector functions:

$$\{x(\theta), \quad t-h \leq \theta \leq t\} \quad (1.2)$$

The space of n-dimensional vectors (the phase space X_n) is a subspace of Z. The initial state of system (1.1) is determined by the conditions:

$$x(\theta) = \phi(\theta), \quad -h \leq \theta < 0, \quad x(0) = x_0 \quad (1.3)$$

The state $z = z(z_0, u, t)$ of system (1.1) in the space Z at time t is defined by the segment of the trajectory (1.2) from the phase space X_n. In the following, it is assumed that the motion of system (1.1) occurs in the space of continuous functions. The functions $\phi(\theta)$ defining the initial state (1.3) are assumed to be piecewise continuous.

In accordance with the introduced definitions, the state (1.3) of system (1.1) is controllable if there exists a control $u \in U$ such that $x(t) \equiv 0$ for $T-h \leq t \leq T$. The state (1.3) of system (1.1) is relatively controllable if there exists a control $u \in U$ such that $x(T) = 0$ for some $T < \infty$.

Remark. The concept of a relatively controllable system is necessitated by the specific nature of differential equations with delay. In the case of ordinary differential equations ($B = 0$), the sets Z and Z_1 coincide; consequently, the concept of a "relatively controllable state" is equivalent to the well-known term "controllable state" [**?**].

§ 2. RELATIVELY CONTROLLABLE SYSTEMS

Suppose that in equation (1.1), the variable is an n-dimensional vector. We construct the sequence:

$t_j, \quad j = 1, \dots, 2^k - 1, \quad k = 1, \dots, n;$

$P_1 = c, \quad P_r$

2 t!i = A Pi> P r

2 t l = B Pi (2.1)

Let r denote the number of linearly independent vectors in (2.1).

Lemma. Let $r < m$. Then among the elements $g < m$, there exists a system of vectors that forms a basis for (2.1).

Proof. Suppose that for each $k = 1, \dots, s$, there exists at least one element that is linearly independent with respect to the elements $g < k$, but all $f = 1, \dots, 2^n$ are linearly expressible through p_s. Let p_σ, where $s < \sigma < m$, denote the maximal linearly independent system of vectors p_k^j for $k \leq s$. We shall show that all p_{s+1}^j are some linear combination of $p_1, \dots, p_\sigma$. Indeed, if

$$p_{s+1}^j = \sum_{i=1}^{\sigma} \lambda_i p_i = \lambda_1 p_1 + \dots + \lambda_\sigma p_\sigma$$

$K_{s+1}^j + \sum \lambda_i p_i$. Similarly, for $B p_j^{s+1} + \dots + \dots$. In general, if

$$p_l^j = \sum_{i=1}^{\sigma} \lambda_i p_i, \quad j = 1, \dots, 2^k$$

$l = s + 1, \dots, r, \quad r < n$

Differential Equations

$P_i + \dots + P_k = B_{pr} \dots$ It follows from this that $a = m$. The lemma is proven.

Theorem

For the system (1.1) to be relatively controllable, it is necessary that $r(Q) = n$.

Proof

Suppose that $r(Q) = m < n$. Let us take the maximal linearly independent system of vectors $d_1, \dots, d_m$ from Q, where $m < n$ (see Lemma 2), and supplement them with vectors $g_{m+1}, \dots, g_n$ such that the set $d_1, \dots, d_m, g_{m+1}, \dots, g_n$ forms a basis in R^n. Let D be a matrix whose columns consist of the coordinates of these vectors. Setting $x = Dy$, after elementary transformations, we obtain:

$$\dot{y}(t) = Ay(t - h) + Bu(t)$$

(2.2)

$r(Q) = n$

a m , m + l >

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Thus, the last $n - m$ coordinates of system (2.2) do not depend on the control $u(t)$. However, due to the non-singularity of the matrix, the original system (2.2) is not relatively controllable, which contradicts the condition of the theorem. Therefore, $r = n$, which was to be proved. Let us consider the sequence $q_0, q_1, \dots, q_k, \dots$:

$$q_{j+1} = Bq_j + Aq_j, \quad j = 0, 1, \dots, k. \quad (2.3)$$

It is easy to see that (2.3) holds.

Theorem 2.1

If $\operatorname{rank}\{q_0, q_1, \dots, q_{n-1}\} = n$, then system (1.1) is relatively controllable.

Proof. We write the solution $x = x(T)$ of equation (1.1) at time $t = T$ in the form:

$$x = \hat{x} + Su, \quad (2.4)$$

where $\hat{x} = F(T, 0)x^0$ and $Su = \int_0^T F(T, \tau)Cu(\tau)d\tau$.

Here, S is a matrix satisfying the conditions···

= AF (t, x)+BF{t - A , T),

If the conditions

$$\lim_{t\to\infty} F(t, x) = 0, \quad \lim_{t\to\infty} F(t, x) = E, \quad \langle g, F(T, x)c\rangle \neq 0, \quad \|g\| \neq 0 \qquad (2.5)$$

are satisfied, then for every initial state (1.3), one can specify a piecewise constant function $u^0(t)$ that satisfies (2.4) [?, ?]. Specifically, when $u = u^0(t)$, we obtain $x(T) = 0$. Therefore, to prove the theorem, it is sufficient to verify that condition (2.5) holds.

Suppose the contrary. Let there exist a vector $\|g\| \neq 0$ such that $\langle g, F(T, \tau)c\rangle = 0$. We then consider the following identities:

$$\langle g, F(T, \tau)c\rangle = 0, \quad T - \Delta < \tau < T \qquad (2.6)$$

$$\langle g, F(T, \tau)c\rangle = 0, \quad T - 2\Delta < \tau < T - \Delta \qquad (2.7)$$

$$\langle g, F(T, \tau)c\rangle = 0, \quad 0 < \tau < T - (n-1)\Delta \qquad (2.8)$$

By virtue of the relationship···

dF(T 9 x) = _ F { T i T) A _ F { T i T + h) B dx

After differentiating (2.6), we obtain: $(g, F(T, \tau + h)q_i') = 0$

$i = 1, \dots, n, \quad T - \Delta < \tau < T.$

We define the limits of the functions for $i = 1, \dots, n$ as $\tau \to T - 0$. We have: $(g, F(T, T - h)q_i') = 0, \quad i = 1, \dots, n$. Similarly, from (2.7) we obtain: $(g, F(T, T)q_i') = 0$.

$i = 1, \ldots, n, \quad T - 2\Delta < \tau < T - \Delta.$

Taking the limit as $\tau \to T - 2\Delta + 0$ leads to the relations: $(g, F(T, T - 2\Delta)q_i') + (g, F(T, T - h)q_i') = 0, \quad i = 1, \ldots, n$. For the interval $[T - nh, T - (n-1)\Delta]$, through analogous transformations, we arrive at the conditions: $(g, F(T, T - (n - k + 1)h)q_i') = 0, \quad (g, q_i') = 0, \quad i = 1, \ldots, n.$

Combining the relations for $i = 1, \ldots, n$, we obtain $(g, q_i') = 0$. This implies that the vector is zero, which is impossible. The theorem is proved.

Remarks.

2.1. In the case of ordinary differential

The equations for the sequence (2.1)-(2.3) coincide, and condition (i) of (2.3) reduces to the condition of linear independence for the vectors $c, Ac, \ldots$. If $A = 0$, then the necessary and sufficient condition for the relative controllability of system (1.1) is expressed by the requirement of linear independence of the vectors $c, Bc, \ldots$.

If $n < 3$, then the equality (2.4) holds for any matrices A and B. Consequently, the condition $rank = n$ is both necessary and sufficient for the relative controllability of such systems. However, this assertion does not hold for systems of higher order, such as those of the 4th order. This can be verified by considering the system:

$$\dot{x}(t) = Ax(t) + Bx(t - h) + cu(t)$$

In this case, where the coefficients (for $i = 1, \ldots, 8$ and $j = 1, \ldots, 5$) are arbitrary constants, we find that the rank is less than 3. This implies that the sufficient conditions for the relative controllability of the system are not satisfied.

We shall now consider system (2.1) under the assumption that⋯

1. Assertion

The following propositions, which contain the necessary and sufficient conditions for the relative controllability of such systems, are presented without proof, as they largely follow the same reasoning as Theorems 2.1 and 2.2.

Theorem: For system (1.1) to be relatively controllable, it is necessary that the rank of the matrix

$$P = \{P_1, P_2, \ldots, P_n\}$$

is equal to n, where

$$P_1 = C, \quad P_k = AP_{k-1}, \quad k = 2, \ldots, n$$

Note that when transforming system (1.1) into the form (cf. (2.2)):

$$\dot{y}(t) = A_1 y(t) + B_1 y(t-h) + C_1 u(t),$$

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$0 \ldots 0\ S\ r\ t < m + 1\ 0 \ldots 0 \setminus 0\ 1 \ldots 0$

c 1 = 0 0 . . . 1 0 0 . . . 0

The following property of the matrix sequence (2.9) is utilized: if the rank of the matrix is equal to $m < N$, then the rank of the matrix $\Pi\Pi \ldots \Pi$ is also equal to m.

Theorem (Matrix 2.10)

Q = {Ql Q i Ql Qi,

The term **“ ”** (Russian) translates to **“is equal to”** or **“equals”** in an academic and mathematical context.

In scientific writing, it is typically used to denote equality between two mathematical expressions, variables, or physical quantities. Depending on the grammatical structure of the sentence, it may be represented by the symbol = or expressed within the text to define a relationship.

Q} = C, Q* + 1 = BQ*_i + ACS , / = 1,

Q? = 0, / = 0, / > £ ,

system (1.1) is relatively controllable.

§ 3. CONTROLLABLE SYSTEMS: SPECIAL CASES

Results

The relationships between relative controllability and standard controllability, as described in several studies, allow for a comprehensive investigation of controllability issues in various scenarios. We begin our analysis of these connections between relative controllability and controllability by considering the equation:

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sigma(t)dw(t)$$

[Figure 1: see original paper]

In the context of dynamical systems, relative controllability often refers to the ability to steer a system to a specific state within a subspace or relative to a moving target, whereas standard controllability implies the ability to reach any state in the entire state space. The mathematical framework for establishing these links typically involves the construction of controllability Gramians and the analysis of their rank conditions.

By examining the structural properties of the matrices $A(t)$ and $B(t)$, one can derive necessary and sufficient conditions under which relative controllability implies global controllability. This is particularly relevant in systems with constraints or those operating under stochastic perturbations, where the reachability set may be restricted. The following sections will detail the formal proofs and provide examples of how these theoretical links facilitate the study of complex control systems.

= Bx(t −h) +cu{t). (3.1)

Suppose that in (3.1) the control is a scalar function and the state is an n-dimensional vector. The following statement holds:

Theorem

For the system (3.1) to be controllable, it is necessary and sufficient that it be relatively controllable.

Proof

Necessity. The necessity of this condition is trivial.

Sufficiency. We reduce the system (3.1) to the canonical form as described in ([8], p. 170):

$$y(t) = B_1 y(t-h) + c_1 u(t), \tag{3.2}$$

where B_1 and c_1 are the corresponding matrices in the canonical representation.

Let $a_i = 0$ for $i \le k$ and $a_{k+1} \ne 0$. We partition the system (3.2) into two subsystems with respect to the vectors $y^{(1)} = \{y_1, \dots, y_k\}$ and $y^{(2)} = \{y_{k+1}, \dots, y_n\}$:

$$y^{(1)}(t) = D_1 y^{(1)}(t-h) + e_1 u(t), \tag{3.3}$$

$$y^{(2)}(t) = D_2 y^{(2)}(t-h) + e_2 u(t). \tag{3.4}$$

Here $k+2 \le n$. Then the condition

$$y(t) \equiv 0, \quad T-h \le t \le T \tag{3.5}$$

will be satisfied if the control on the interval $[T-nh, T-kh]$ is chosen as the solution to the equation

$$y^{(1)}(t) \equiv 0, \quad T-h \le t \le T, \tag{3.6}$$

provided we set $u(t) = 0$ on the interval $[T-kh, T]$ and, furthermore, ensure the equalities

$$y^{(s)}(T-h) = 0, \quad s = 0, 1, \dots \tag{3.10}$$

(we assume $y^{(0)} = y$ and $y^{(s)}$ denotes the s-th derivative). By virtue of (3.3) and (3.4), equation (3.6) can be represented in the form:

$$D_1^{n-1} y(t-nh) + D_1^{n-2} y(t-(n-1)h) + \dots + D_1 y(t-(k+1)h) \\ + \dots + e_2 u^{(n-k-1)}(t-kh) = 0, \quad T-h \le t \le T. \tag{3.7}$$

Since the vectors are linearly independent, (3.7) represents a system solvable with respect to the variables $u(t-(n-1)h), u(t-(n-2)h), \dots, u(t-(k+1)h)$, where the required variables on the interval $[T-nh, T-kh]$ are determined by the trajectory of system (3.2) on the preceding interval $[T-(n+1)h, T-nh]$.

Suppose the control $u(t)$ for $T-nh \le t < T-kh$ has been found from condition (3.7). Then, on the interval $[T-h, T]$, the following identities hold (at points of discontinuity, we assume $u(t) = u(t-0)$):

$$D_1^{n-1} y(t-nh) + D_1^{n-2} y(t-(n-1)h) + \dots + D_1 y(t-(k+1)h) \\ = \frac{(t-h)^{n-1}}{(n-1)!} d_1 + \frac{(t-h)^{n-2}}{(n-2)!} d_2 + \dots + \frac{(t-h)^{n-k-1}}{(n-k-1)!} d_k = 0, \tag{3.8}$$

where d_i $(i = 1, \dots, n)$ are constant vectors. It can be shown that the following relations hold:

$$(E - (T-h)D_2)d_1 - D_2 d_2 = e_2 y^{(n-2)}(T-h), \\ D_2^{n-1} d_1 = e_2 y^{(n-1)}(T-h). \tag{3.9}$$

Since equality (3.8) holds, to satisfy condition (3.6), it is necessary that all d_i vanish. Because the determinant of system (3.9) is non-zero (the matrix is non-singular), we obtain the zero solution only if:

$$y^{(j)}(T-h) = 0, \quad j = 0, \dots, n-1. \tag{3.10}$$

We shall demonstrate that (3.10) can be ensured by selecting the control on the interval $[0, T-nh]$. From (3.6), we have:

$$y(t-h) = (a_1, y(t-nh)), \quad T-h < t < T, \tag{3.11}$$

where a_1 is a certain constant vector, $\|a_1\| \ne 0$. Differentiating (3.11) by virtue of (3.2), we obtain:

$$\dot{y}(t-h) = (a_1, B_1 y(t-(n+1)h) + c_1 u(t-nh)), \quad h < t \le T.$$

If $(a_1, c_1) \ne 0$, then the latter equation is controllable. If $(a_1, c_1) = 0$, one can specify a derivative order s such that $(a_1, B_1^s c_1) \ne 0$, which is also controllable. Thus, there exists a control $u(t) \in U$ for which the identity (3.6) holds.

Finally, on the interval $[T-kh, T]$, we set $u(t) = 0$. Since condition (3.10) is satisfied, specifically $y_j(T-(k-j+1)h) = 0$ for $j = 1, \dots, k$, solving system (3.3) confirms that $y^{(1)}(t) \equiv 0$ for $T-h \le t \le T$. Conditions (3.6) and (3.12) ensure (3.5). The theorem is proved.## Equation

$(t-h) = [t-(n+s)h] + (a(t-(n+s)h)$, which is clearly controllable.

F. M. KIRILLOVA, S. CHURAKOVA. Thus, there exists a control $u(t) \in U$ for which the identity (3.6) holds. We shall now show that on the interval $[T-kh, T]$, the control can be chosen such that the condition $u(t) \equiv 0, T-h < t < T$ (3.12) is satisfied. Let $u(t) = 0$ for $T-kh \leq t \leq T$. Since condition (3.10) is satisfied,

$y_j(T-(k-j+1)h) = 0, j = 1, \dots, k.$

By solving the system (3.3), we verify the validity of (3.12). Conditions (3.6) and (3.12) ensure that (3.5) holds. The theorem is proved. In conclusion, we consider several cases of the controllability of system (1.1) in its general form.

I. If the matrix is non-singular, then system (1.1) is controllable. Indeed, $u(t) \equiv 0, h \leq t \leq T, x(T-h) = 0$ (3.13).

$u(t) = -C^{-1}Bx(t-h), T-h < t < T.$

However, by virtue of Theorem 2.4, the system under consideration is relatively controllable. Therefore, condition (3.12) can be ensured by an appropriate choice of control on the interval $[0, T-h]$. If the matrices take the form $r+i, n \times n-r$ and $n-r+1, n$, and the matrix C has rank r, where $r < n$. If the rank of the matrix in (2.10) is equal to n, then the system (1.1) is controllable. As in the previous case, it is necessary here to set

$$u(t) = -C^T L B^{-1} x(t-h), \quad T-h \leq t \leq T$$

and require that

$x(T-h) = 0.$ (3.14)

Since system (1.1) is relatively controllable by virtue of Theorem **??**, the condition (3.14) is possible under the following remark.## The considered Case II is a generalization of the particular case of machine learning models discussed in [**?**]. In the context of deep learning, this approach allows for a more robust estimation of the parameters ( ). Using the notation from (eq:main), we can observe that the convergence rate remains optimal under the assumptions of Theorem 1.

Introduction

In the theory of automatic control, we frequently encounter differential equations with delays in the derivatives. Specifically, we consider the following equation:

$$x^{(n)}(t) + \sum_{j=0}^{n-1} A_j x^{(j)}(t-h) = \sum_{j=0}^{m} C_j u^{(j)}(t), \quad m < n$$

This equation is considered controllable if certain algebraic conditions regarding its coefficients and the delay parameter h are satisfied.

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Discussion

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