The asymptotic behavior of solutions of certain nonlinear differential equations

Authors: I. A. Toroshelidze

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Abstract

The differential equation

$$u^{(n)} + a(x)u = f(x, u, u', \dots, u^{(n)}) \tag{1}$$

is considered. A series of sufficient conditions is established under which equation (1) has solutions with the same asymptotic form as the solutions of the equation $u^{(n)} + a(x)u = 0$.

Bibliography: 4 items.

Full Text

Preamble

In this section, we consider the asymptotic behavior of solutions to n-th order differential equations of the form:

$$u^{(n)} + a(x)u = f(x, u, u', \dots, u^{(n-1)}) \tag{1.1}$$

where the function $f(x, y_1, \dots, y_n)$ satisfies certain growth conditions relative to the linear part of the equation. We are interested in establishing the existence and uniqueness of solutions $u(x)$ that exhibit specific asymptotic properties as $x \to \infty$.

Consider the related linear equation:

$$v^{(n)} + a(x)v = 0 \tag{1.2}$$

Let $y = (y_1, \dots, y_m)$ and let $L(x, t, y)$ be an n-dimensional vector function. We define the operator A_k acting on the space S_k of continuous functions. Under the

Carathéodory conditions, we assume there exist functions $\phi_j(x,t)$ and $\psi_j(x,t)$ such that:

$$\|K_j(x,t,y)\| \leq \phi_j(x,t), \quad \|L_j(x,y,z)\| \leq \psi_j(x,t) \tag{1.3}$$

Furthermore, we assume the following integrability conditions hold for $x > a > 0$:

$$\int_a^\infty \phi_1(x,t)dt < r_1, \quad \int_a^\infty \phi_2(x,t)dt < r_2 \tag{1.4}$$

$$\int_a^\infty \psi_1(x,t)dt + \int_a^\infty \psi_2(x,t)dt < \infty \tag{1.5}$$

We define the operator $A_k y(x)$ for $y(x) \in S_k$ as:

$$y(x) = \int_a^x L_1(t,y(t),z_1(t))dt + \int_a^x L_2(t,y(t),z_2(t))dt \tag{1.6}$$

where $z_1(t)$ and $z_2(t)$ are integral terms involving the kernels K_1 and K_2. Using the estimates in (1.3) and (1.4), one can show that A_k maps the set S_k into itself. To prove the existence of a fixed point, we examine the continuity and compactness of the operator A_k. Specifically, we consider a sequence $\{y_j(x)\} \subset S_k$ and evaluate the limit:

$$\lim_{j\to\infty} \|A_k y_j(x_2) - A_k y_j(x_1)\| = 0 \tag{1.8}$$

By applying the Schauder fixed-point theorem, we establish the existence of a solution $y_k(x)$ to the integral equation (1.11) on the interval $[a, a+k]$. By extending this interval and considering the limit as $k \to \infty$, we obtain a solution $y(x)$ defined for all $x \geq a$ that satisfies the required asymptotic conditions (1.5).

Section 2. Case with $a(x) = 0$

We first analyze the case where $a(x) = 0$. Suppose the nonlinear term satisfies the following inequality:

$$|x^{n-k} f(x, x^{k-1}y_1, \dots, x^{k-n}y_n)| \leq \phi(x, \|y\|) \tag{2.1}$$

for $1 \leq k \leq n$. We consider the differential equation:

$$u^{(n)} = f(x, u, u', \dots, u^{(n-1)}) \tag{2.2}$$

with the asymptotic boundary conditions:

$$u^{(j-1)}(x) = c_j x^{k-j} + o(x^{k-j}), \quad j = 1, 2, \dots, n \tag{2.3}$$

where c_j are constants. By defining appropriate kernels K_v and operators L_v as shown in (2.6) and (2.7), we can transform the differential equation into an equivalent system of integral equations.

Theorem 2.1. If the function $\phi(x, X)$ is monotonic and satisfies the integrability condition:

$$\int_a^\infty \phi(t, k\|c\| + 1)dt < \infty \tag{2.8}$$

then there exists a solution to (2.2) satisfying the asymptotic representation (2.3).

Theorem 2.2. For the linear case where $f(x, u, \dots) = \sum b_j(x)u^{(j-1)}$, if the coefficients satisfy:

$$x^{n-j}|b_j(x)| \le \phi(x) \tag{2.11}$$

then the equation $u^{(n)} = \sum b_j(x)u^{(j-1)}$ has a solution satisfying (2.13).

Theorem 2.3. Consider the inequality $|f(x, y_1, \dots, y_n)| \le \phi(x, |y_1|)$. Let $v(x)$ be a solution to the comparison equation. If the integral of ϕ converges, then the solution $u(x)$ to (2.2) is bounded by $v(x)$ as $x \to \infty$. This allows us to establish the stability of the asymptotic behavior (2.13) under perturbations of the initial conditions or the functional form of f.

Theorem 2.4. If a solution $u(x)$ to (2.2) exists on $[x_0, \infty)$ and satisfies the condition $x^{j-n}|u^{(j-1)}(x)| < X$, then it must satisfy the asymptotic property (2.13) provided the conditions of Theorem 2.1 hold.

Section 3. General Case for $a(x)$

In the general case, we assume $a(x)$ is a continuous function. Let $\sigma_k(x)$ be the fundamental system of solutions to the linear equation (1.2). We define:

$$\mu(x) = |a(x)|^{\frac{2-n}{2n}} \exp\left(\int |a(t)|^{1/n} dt\right) \tag{3.2}$$

Theorem 3.1. Suppose f satisfies:

$$|f(x, G_{k1}y_1, \dots, G_{kn}y_n)| \le \phi(x, \|y\|) \tag{3.3}$$

where G_{kj} are weight functions related to the linear solutions. Then equation (1.1) has a solution $u_k(x)$ such that:

$$u_k^{(j-1)}(x) = c\sigma_{kj}(x) + o(\sigma_{kj}(x)) \tag{3.5}$$

where σ_{kj} represents the j-th derivative of the k-th fundamental solution.

Theorem 3.2. If the nonlinear term f satisfies the growth condition (3.18), then any solution $u(x)$ to (1.1) can be represented as a linear combination of the fundamental solutions of the linear part plus a vanishing error term:

$$u^{(j-1)}(x) = \sum_{k=1}^{n} [c_k\sigma_{kj}(x) + o(\sigma_{kj}(x))] \tag{3.19}$$

This result is obtained by applying the method of variation of parameters and utilizing the estimates for the Wronskian $W(x)$ of the system. The convergence of the integrals in (3.20) ensures that the coefficients c_k approach stable limits as $x \to \infty$.

Theorem 3.3. Under the conditions of Theorem 3.2, the mapping between the initial data at x_0 and the asymptotic coefficients c_k is continuous. Furthermore, if the integral condition (3.24) is satisfied, the asymptotic behavior (3.19) is unique for a given set of coefficients.

Note: Figure translations are in progress. See original paper for figures.

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