ON OPTIMAL PROCESSES IN TWO-PARAMETER DISCRETE SYSTEMS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 519.3

MATHEMATICS

O. V. VASIL' EV, F. M. KIRILLOVA

ON OPTIMAL PROCESSES IN TWO-PARAMETER DISCRETE SYSTEMS

(Presented by Academician L. S. Pontryagin on 25 VIII 1966)

1. Let, at the nodes $(mh, n\tau)$ of the rectangle

$$D = \{mh, n\tau : \ m = 0, 1, \dots, M, \ n = 0, 1, \dots, N\},$$

the relation between the q-dimensional vector $x = x(m, n)$ and the r-dimensional control vector $u = u(m, n)$ be given by the equation

$$R_{h\tau}(x(m, n)) = f(x, R_h, R_\tau, u, m, n) \tag{1}$$

and by the initial conditions

$$x(0, n) = \gamma^n, \quad n = 0, 1, \dots, N; \qquad x(m, 0) = \beta^m, \quad m = 0, 1, \dots, M, \tag{2}$$

where

$$\begin{aligned} R_{h\tau}(x(m, n)) = [x(m+1, n+1) - x(m+1, n) - x(m, n+1) + \\ + x(m, n)]/h\tau, \\ R_h = R_h(x(m, n)) = [x(m+1, n) - x(m, n)]/h, \\ R_\tau = R_\tau(x(m, n)) = [x(m, n+1) - x(m, n)]/\tau, \end{aligned}$$

$f(x, R_h, R_\tau, u, m, n)$ is uniquely defined on

$$E^q \times E^q \times E^q \times E^r \times E^1 \times E^1,$$

is continuous, and has there continuous second derivatives with respect to x, R_h, R_τ.

Define the class of **admissible controls** ([1,2]) by requiring that $u(m, n) \in U$, where U is a given subset of E^r. On the trajectories (1)–(2) define the functional

$$S(u) = (c \cdot x(M, N)).$$

Problem. Find $u^0(m, n) \in U$ $(m = 0, 1, \dots, M-1; \ n = 0, 1, \dots, N-1)$ such that

$$S(u^0) = \min_{u \in U} S(u). \tag{3}$$

A solution u^0 of problem (3) will be called an **optimal control**.

2. To the control $u(m,n) \in U$ there corresponds a solution $x(m,n)$; to the control

$$u(m,n) + \Delta u(m,n) \in U$$

there corresponds a solution $x(m,n) + \Delta x(m,n)$, and

$$\begin{aligned} R_{h\tau}(\Delta x(m,n)) = f(x+\Delta x, R_h + \Delta R_h, R_\tau + \Delta R_\tau, u + \Delta u, m, n) \\ - f(x, R_h, R_\tau, u, m, n), \end{aligned} \tag{4}$$

$$\Delta x(m,0) = \Delta x(0,n) = 0,$$

where

$$\Delta R = R(x(m,n) + \Delta x(m,n)) - R(x(m,n)) = R(\Delta x(m,n)).$$

For an arbitrary sequence of q-dimensional vectors $p(m,n)$,

$$m = -1, 0, \dots, M-1; \quad n = -1, 0, \dots, N-1,$$

the following equality holds:

$$\begin{aligned} \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(p(m,n)\cdot R_{h\tau}(\Delta x(m,n)))h\tau = &\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(\overline{R}_{h\tau}(p(m,n))\cdot \Delta x(m,n))h\tau \\ &- \sum_{m=0}^{M-1}(\overline{R}_h(p(m,N-1))\cdot \Delta x(m,N))h \\ &- \sum_{n=0}^{N-1}(\overline{R}_\tau(p(M-1,n))\cdot \Delta x(M,n))\tau \\ &+ (p(M-1,N-1)\cdot \Delta x(M,N)), \end{aligned} \tag{5}$$

where

$$\overline{R}_{h\tau}(p(m,n)) = [p(m,n) - p(m-1,n) - p(m,n-1) + p(m-1,n-1)]/h\tau,$$

$$\overline{R}_h(p(m,n)) = [p(m,n) - p(m-1,n)]/h, \qquad \overline{R}_\tau(p(m,n)) = [p(m,n) - \\ - p(m,n-1)]/\tau.$$

Put

$$H(x,p,R_h,R_\tau,u,m,n) = (p(m,n)\cdot f(x,R_h,R_\tau,u,m,n))$$

and choose $p(m,n)$ from the equations

$$\overline{R}_{h\tau}(p(m,n)) = \partial H(x,p,R_h,R_\tau,u,m,n)/\partial x -$$
$$-\overline{R}_h(\partial H(x,p,R_h,R_\tau,u,m,n)/\partial R_h) - \overline{R}_\tau(\partial H(x,p,R_h,R_\tau,u,m,n)/\partial R_\tau) \quad (6)$$

with boundary conditions

$$p(M-1,N-1) = -c, \quad (7)$$
$$\overline{R}_h(p(m,N-1)) = -\partial H(x,p,R_h,R_\tau,u,m,N-1)/\partial R_\tau,$$
$$\overline{R}_\tau(p,(M-1,n)) = -\partial H(x,p,R_h,R_\tau,u,M-1,n)/\partial R_h.$$

From (4), (6), (7) and identity (5) we obtain

$$-(c\cdot\Delta x(M,N)) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h\tau[H(x,p,R_h,R_\tau,u+\Delta u,m,n) -$$
$$-H(x,p,R_h,R_\tau,u,m,n)] + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h\tau[H(x+\Delta x,p,R_h+\Delta R_h,R_\tau+\Delta R_\tau,$$
$$u+\Delta u,m,n) - H(x,p,R_h,R_\tau,u+\Delta u,m,n) -$$
$$-\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h\tau\left[\left(\frac{\partial H(x,p,R_h,R_\tau,u,m,n)}{\partial x}\cdot\Delta x(m,n)\right)+\right.$$
$$\left.+\left(\frac{\partial H(x,p,R_h,R_\tau,u,m,n)}{\partial R_h}\cdot\Delta R_h(m,n)\right)+\left(\frac{\partial H(x,p,R_h,R_\tau,u,m,n)}{\partial R_\tau}\cdot\Delta R_\tau(m,n)\right)\right].$$

Hence

$$\Delta S \equiv S(u+\Delta u) - S(u) \equiv (c\cdot\Delta x(M,N)) =$$

$$= -\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h\tau[H(z,p,u+\Delta u,m,n) - H(z,p,u,m,n)] - \eta, \qquad \eta+\eta_1+\eta_2,$$

$$\eta_1 = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h\tau\left(\left[\frac{\partial H(z,p,u+\Delta u,m,n)}{\partial z} - \frac{\partial H(z,p,u,m,n)}{\partial z}\right]\cdot\Delta z(m,n)\right),$$

$$\eta_2 = \frac{1}{2}h\tau\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(\Delta z(m,n)\times$$

$$\times \frac{\partial^2 H(z+\theta\Delta z, p, u+\Delta u, m, n)}{\partial z^2}\Delta z(m,n)\Bigg), \qquad 0 \leqslant \theta(m,n) \leqslant 1.$$

Here $z = \{x, R_h, R_\tau\}$ is a $3q$-dimensional vector.

For the variation

$$\Delta^* u(m,n) = \begin{cases} u^* - u(k,l), & m = k,\ n = l, \\ 0, & m \neq k,\ n \neq l, \end{cases}$$

we have

$$\Delta^* S \equiv S(u + \Delta^* u) - S(u) = -h\tau\{[H(z,p,u^*,k,l) - H(z,p,u,k,l)] - \eta^*\},$$

$$\eta^* = \frac{1}{2}\sum_{m=k+1}^{M-1}\sum_{n=l+1}^{N-1}\left(\Delta z(m,n)\cdot\frac{\partial^2 H(z+\theta\Delta z, p, u, m, n)}{\partial z^2}\Delta z(m,n)\right), \qquad \eta_1^* = 0.$$

It is not difficult to show that

$$|\eta^*| \leqslant \frac{1}{2}(h\tau)^2\|f(z,u^*,k,l) - f(z,u,k,l)\|^2 \times \\ \times \sum_{m=k+1}^{M-1}\sum_{n=l+1}^{N-1} L_{m,n}\left\|\frac{\partial^2 H(m,n)}{\partial z^2}\right\| = \overline{\eta}. \tag{8}$$

3. Let $f(z,u,m,n)$ be differentiable with respect to u. Introduce the set $a(\alpha,k,l)$:

$$a(\alpha, z, p, k, l, u(k,l), \ldots, u(M-1,N-1), h, \tau) = \{u^* : |\eta^*| \leq \alpha\}$$

and the quantity $\delta_u H$

$$H(z,p,u+\Delta u,m,n) - H(z,p,u,m,n) = \delta_u H(z,p,u,m,n) + o(\|\delta u\|).$$

Definition. The control u at the node (k,l) satisfies the quasimaximum condition with number $\mu(k,l)$ and set $\omega(k,l)$, if:

1) $H(z,p,u,k,l) \geq H(z,p,u^*,k,l) - \mu(k,l)$ for all $u^* \in \omega(k,l)$;
2) $\delta_u H \leq 0$ if U is convex,
3) $\delta_u H = 0$ at interior points of U.

Theorem 1. *The optimal control $u^0 = u^0(m,n)$ for problem (3) satisfies the conditions:*

1) *quasimaximum with $\mu(k,l) = \alpha,\ \omega(k,l) = a(\alpha,k,l) \cap U,\ k = 0,1,\dots,M-2;\ l = 0,1,\dots,N-2;$*

2)

$$H(z,p,u^0,k,l) \geq H(z,p,u,k,l), \qquad u \in U \tag{9}$$

for $k = M-1;\ l = 0,\dots,N-1$ and $k = 0,\dots,M-1;\ l = N-1$.

Remark. For convenience of computation, the set $a(\alpha,k,l)$ may be replaced by the set $a'(\alpha,k,l)$:

$$a'(\alpha,k,l) = \{u^* : |\overline{\eta}| \leq \alpha\},$$

where $\overline{\eta}$ is some upper estimate for η^*, for example (8).

From the definition of the set $a(\alpha,k,l)$ it follows that the local-maximum principle is valid at the node (k,l) if the set $a(0,k,l)$ contains points distinct from $u^0(k,l)$.

Thus, the quasimaximum condition singles out some set $U^* \subset U$ of points $u(k,l)$ suspected of optimality. The narrowing of the set U^* is carried out by varying α and by the local conditions $\delta_u H \leq 0,\ \delta_u H = 0$.

Let $h \to 0,\ \tau \to 0$; then the number α^* from $a'(\alpha^*,k,l) \supseteq U$ tends to zero, and the quasimaximum condition passes into the condition of the absolute maximum, with respect to u, of the function H.

4. For the linear variant

$$\begin{aligned} R_{h\tau}(x(m,n)) = A(m,n)x(m,n) + B(m,n)R_h(x(m,n)) + \\ + C(m,n)R_\tau(x(m,n)) + \varphi(m,n,u(m,n)) \end{aligned} \tag{10}$$

of equation (1), we have $\eta = 0$.

Theorem 2. *In order that the control $u^0 \in U$ for (10) be optimal, it is necessary and sufficient that condition (9) be fulfilled at the nodes (m,n), $m = 0,1,\dots,M-1;\ n = 0,1,\dots,N-1$.*

5. For the case

$$R_{h\tau}(x(m,n)) = A(m,n,u(m,n))x(m,n) + B(m,n,u(m,n))R_h(x(m,n)) +$$

$$+ C(m,n,u(m,n))R_\tau(x(m,n)) + \varphi(m,n,u(m,n))$$

it is not difficult to see that $\eta^* = 0$.

Theorem 3. *If $u^0 = u^0(m,n)$ is an optimal control, then at each node (k,l), $k = 0, 1, \dots, M-1$; $l = 0, 1, \dots, N-1$, (9) is fulfilled.*

6. Let U be convex and let equation (1) have the form

$$R_{h\tau}(x(m,n)) = \sum_{i=1}^{r} u_i(m,n) \cdot f_i(x, R_h, R_\tau, m, n) + g(x, R_h, R_\tau, m, n).$$

Theorem 4. *For the optimal control u^0, condition (9) is fulfilled at all nodes of the rectangle D.*

Ural Polytechnic Institute
named after S. M. Kirov

Received
10 VI 1966

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.