SOLVABILITY CONDITIONS FOR SOME BOUNDARY-VALUE PROBLEMS FOR AN ORDINARY LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

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Abstract

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Yu. V. KOMLENKO

SOLVABILITY CONDITIONS FOR SOME BOUNDARY-VALUE PROBLEMS FOR AN ORDINARY LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

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1. In papers ([1–8, 11, 12, 14, 15]) conditions for the existence and preservation of sign of the Green' s function of linear boundary-value problems are considered, and the role of such conditions in the investigation of a number of questions in the qualitative theory of differential equations and mathematical physics is noted. For the equation

$$x'' + q(t)x' + p(t)x = f(t), \tag{1}$$

where $q(t)$, $p(t)$, and $f(t)$ are real summable functions on $[a, b]$ of the real variable t, with boundary conditions

$$x(a) = x'(b) = 0, \tag{2}$$

$$x(a) = x(b) = 0, \tag{3}$$

we propose, in terms of q and p, theorems on the existence and preservation of sign of the Green' s function $G(t, s)$. The solution $x(t)$ of the problems under consideration will be represented in the form

$$x(t) = \int_a^b G(t, s) f(s)\, ds.$$

Therefore the condition of preservation of sign of $G(t, s)$ is a condition for the validity of a theorem on a differential inequality, which plays an important role in constructing various estimates ([3–6]), and, on the basis of the results of ([14]), makes it possible to formulate the corresponding existence theorems for nonlinear boundary-value problems. The assertions given in the article are obtained from the "fork principle" ([9–11, 14]).

To shorten the statements of the theorems given below, we assume that q and p are not equivalent to piecewise-constant functions. For the same purpose, denote $p_+ = 2^{-1}(|p| + p)$ and introduce the function of two variables

$$F_n(k,m) = \begin{cases} \sqrt{k-m^2}\,\operatorname{ctg}(2n)^{-1}(b-a)\sqrt{k-m^2}, & \text{for } m^2 < k \le m^2 + n^2\pi^2(b-a)^{-2}, \\ -n(b-a)^{-1}, & \text{for } k = m^2, \\ \sqrt{m^2-k}\,\operatorname{cth}(2n)^{-1}(b-a)\sqrt{m^2-k}, & \text{for } k < m^2, \end{cases}$$

where n is a fixed number.

2. Here we consider the boundary-value problem (1), (2). For arbitrary values of the parameters k, m define the functions $h_1(t)$, $h_2(t)$, $g(t)$ by the formulas

$$h_1(t) = \begin{cases} \lambda^{-1}\sin\lambda t & \text{for } k > m^2, \\ t & \text{for } k = m^2, \\ \lambda^{-1}\operatorname{sh}\lambda t & \text{for } k < m^2; \end{cases} \qquad \lambda = \begin{cases} \sqrt{k-m^2} & \text{for } k > m^2, \\ \sqrt{m^2-k} & \text{for } k < m^2; \end{cases}$$

$$h_2(t) = \begin{cases} 2^{-1}\left(\lambda^{-1}\sin\lambda t + \lambda^{-2}m\cos\lambda t + \lambda^{-2}\sqrt{k}\right) & \text{for } k > m^2, \\ 2^{-1}\left(t - 2^{-1}mt^2 - 2^{-1}m^{-1}\right) & \text{for } k = m^2, \\ 2^{-1}\left(\lambda^{-1}\operatorname{sh}\lambda t - \lambda^{-2}m\operatorname{ch}\lambda t - \lambda^{-2}\sqrt{k}\right) & \text{for } k < m^2; \end{cases}$$

$$g(t) = h_1'(t) - mh_1(t).$$

Theorem 1. Let, for some constants k and m, the inequality

$$h_i(b-a)\int_a^b [p(t)-k]_+\,dt + g^2\left(\frac{b-a}{2}\right)\int_a^b |q(t)-2m|_+\,dt \le g(b-a),$$

hold, where $i = 1$, if $F_{1/2}(k,m) \ge |m|$; $i = 2$, if $F_{1/2}(k,m) < -m$.

Then the Green' s function $G(t,s)$ of problem (1), (2) exists, and $G(t,s) \le 0$ for $a \le t, s \le b$.

Corollary 1. The assertion of Theorem 1 is valid if any one of the following conditions is satisfied:

A. Let $q(t) \le 2m$ almost everywhere on $[a,b]$, $m = \text{const} \ge 0$, and for some constant k the inequality

$$\int_a^b [p(t)-k]_+\,dt \le F_{1/2}(k,m)-m$$

holds.

B. Let $q(t) \le 2m = \text{const}$ almost everywhere on $[a,b]$, and suppose the inequality

$$\int_a^b p_+(t)\,dt \le m\,[\operatorname{cth} m(b-a)-1]$$

holds.

C.

$$\int_a^b q_+(t)\,dt + (b-a)\int_a^b p_+(t)\,dt \le 1.$$

Remark. If $m=0$, then condition B generalizes the estimate following from works (9, 13).

3. Consider the boundary-value problem (1), (3).

Theorem 2. Let $|q(t)| \le 2m = \text{const}$ almost everywhere on $[a,b]$, and let, for some constant k, the inequality

$$\int_a^b [p(t)-k]_+\,dt \le 2\,[F_1(k,m)-m]$$

hold.

Then the Green' s function $G(t,s)$ of problem (1), (3) exists, and $G(t,s) \le 0$ for $a \le t, s \le b$.

The proof of this theorem is based on Corollary 1 and on certain facts from the theory of adjoint problems.

Remark. If $m=0$, then from Theorem 2 we obtain Theorem 4 of work (11).

Corollary 2. Let $|q(t)| \le 2m = \text{const}$ almost everywhere on $[a,b]$. If the inequality

$$\int_a^b p_+(t)\,dt \le 2m\,[\operatorname{cth} 2^{-1}m(b-a)-1],$$

holds, then the Green's function $G(t,s)$ of problem (1), (3) exists, and $G(t,s) \le 0$ for $a \le t, s \le b$.

Remark. If $m = 0$, then Corollary 2 gives the well-known Zhukovskii–Krein estimate.

The assertion of Theorem 2 means that any nontrivial solution of the equation

$$x'' + q(t)x' + p(t)x = 0 \tag{4}$$

has on $[a, b]$ no more than one zero. From the point of view of the question of oscillation of solutions, the following Theorem 3 is a generalization of Theorem 2.

Theorem 3. Let $|q(t)| \leq 2m = \text{const}$ almost everywhere on $[a, b]$, and for some constant k and some $n = 1, 2, \ldots$ the inequa-

inequality

$$\int_a^b [p(t) - k]_+ \, dt \leq 2n \, [F_n(k, m) - m].$$

Then any nontrivial solution of equation (4) has no more than n zeros on $[a, b]$.

For the first time, an estimate of the distance between two neighboring zeros of any nontrivial solution of equation (4), expressed in terms of the maxima of the absolute values of the coefficients, was obtained by Valle-Poussin (1). This estimate was improved by P. Hartman and A. Wintner (1), Z. Opial (2), and G. N. Mil' shtein (12). From Theorem 2 there follows the unimprovable estimate:

Corollary 3. If the coefficients of equation (4) are summable and almost everywhere on $[a, b]$ admit the estimate $|q(t)| \leq 2m$ and $p(t) \leq k$ (m, k are constants), then for the distance h between two neighboring zeros of any nontrivial solution of equation (4) the estimate

$$h \geq \begin{cases} \dfrac{2}{\sqrt{k - m^2}} \operatorname{arcctg} \dfrac{m}{\sqrt{k - m^2}}, & \text{if } m^2 < k \leq m^2 + \dfrac{\pi^2}{(b-a)^2}, \\ \dfrac{2}{m}, & \text{if } k = m^2, \\ \dfrac{2}{\sqrt{m^2 - k}} \operatorname{arccth} \dfrac{m}{\sqrt{m^2 - k}}, & \text{if } k < m^2. \end{cases}$$

holds. The equality sign is attained in that and only that case when $p(t)$ is equivalent to k, $q(t)$ is equivalent to

$$M(t) = \begin{cases} 2m, & \text{if } u \leq t < (a+b)/2, \\ -2m, & \text{if } (a+b)/2 \leq t < b, \end{cases} \qquad h = b - a.$$

4. Let us consider equations (1) and (4) under the assumption that the functions $q(t)$, $p(t)$, and $f(t)$ have period ω.

Theorem 4. If $|q(t)| \leq 2m = \text{const}$ almost everywhere on $[0, \omega]$,

$$\int_0^\omega p(t)\,dt \geq \omega m^2$$

and for some constant k the inequality

$$\int_0^\omega [p(t) - k]_+\,dt \leq 4\,[F_2(k, m) - m] \qquad (b - a = \omega),$$

is satisfied, then there exists a unique solution of equation (1) having period ω.

The proof of Theorem 4 follows from Theorem 3 and from the fact that if all solutions of equation (4) have an infinite number of zeros, but no more than two on each closed interval of length ω, then equation (4) has the unique ω-periodic solution $x(t) \equiv 0$.

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Izhevsk
Mechanical Institute

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Note: Figure translations are in progress. See original paper for figures.