A sufficient test for the stability of the trivial solution of a third-order differential equation with periodic coefficients

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Abstract

This article considers a third-order differential equation:

$$x''' + P_1(t)x'' + P_2(t)x' + P_3(t)x = 0,$$

where $P_1(t)$ and $P_3(t)$ are ω-periodic odd coefficients that are non-negative on the interval $[0, \omega/2]$, and $P_2(t)$ is an ω-periodic even positive coefficient satisfying the condition $P_2(t) \geq \omega|P_3(t)|$. The functions $P_1(t)$, $P_2(t)$, and $P_3(t)$ are defined and continuous for all values of t.

For this equation, a sufficient stability criterion for the trivial solution is obtained in the following form:

$$\begin{cases} u'(\omega) \int_0^\omega P_2(t) \overset{+}{u}''(t)\, dt + \int_0^\omega P_3(t) \overset{+}{u}''[tu'(\omega) - \omega u'(t) + u(\omega)]\, dt < 4, \\ \int_0^\omega \overset{+}{u}''(t)[P_2(t)u'(t) + P_3(t)u(t)]\, dt < 1, \end{cases}$$

where

$$u''(t) = \exp\left(-\int_0^t P_1(\tau)\, d\tau\right), \quad \overset{+}{u}''(t) = \exp\left(\int_0^t P_1(\tau)\, d\tau\right).$$

A specific example is considered, for which a stability region defined by the sufficient condition is constructed in the space of four parameters.

Bibliography: 3.

Full Text

Preamble

In this section, we consider the differential equation of the form:

$$x''' + P_1(t)x'' + P_2(t)x' + P_3(t)x = 0$$

where $x = x(t)$ and the coefficients $P_i(t)$ are periodic functions. Following the methodology established in [**?**], we analyze the fundamental system of solutions $x_1(t), x_2(t), x_3(t)$ satisfying the initial conditions $x_i^{(s-1)}(0) = \delta_{is}$ for $i, s = 1, 2, 3$.

We define the auxiliary functions $u(t)$, $u'(t)$, and $u''(t)$ as follows:

$$u''(t) = \exp\left(-\int_0^t P_1(\tau)d\tau\right)$$

$$u'(t) = \int_0^t u''(\tau)d\tau$$

$$u(t) = \int_0^t u'(\tau)d\tau = \int_0^t dx \int_0^x \exp\left(-\int_0^{x_2} P_1(\tau_2)d\tau_2\right) dx_1$$

Under the assumption that $\int_0^t P_1(\tau)d\tau > 0$, these functions characterize the growth and stability of the solutions.

2. Properties of the Integral Operators

Let $P(t)$ be a function defined on $[0, t]$. We assume $\phi(t) \geq 0$ and $P(t)\phi(t)dt > 0$ on the interval $[0, \infty)$. For $0 < \tau < t$, we define the kernel functions:

$$\Phi(\tau) = \tau u'(\tau) - u(\tau) > 0$$

Given that $u(0) = 0$ and $u'(0) = 0$, it follows from the monotonicity of $u''(\tau)$ that $\Phi(\tau)$ is strictly increasing. Furthermore, we establish the inequality:

$$u'(\tau)[u(t) - u(\tau)] > u'(\tau) - u(\tau) - tu'(\tau) + u(t)$$

This relationship is critical for bounding the integral terms in the subsequent stability analysis.

3. Constraints on the Coefficients

We assume the following conditions on the coefficients $P_i(t)$ for $t \in [0, \infty)$: - (a) $P_1(t)$ is such that $u''(t)$ is well-defined and integrable. - (b) $P_2(t) > 0$ for $t < \omega/2$ and $P_2(t)$ remains bounded. - (c) $P_3(t) < 0$ on specific sub-intervals, satisfying $P_2(t) + P_3(t)t > 0$. - (d) $P_2(t)u'(t) + P_3(t)u(t) > 0$.

These conditions ensure that the operator $F_{ji}(t)$, defined by the iterative relation

$$F_{ji}(t) = P_2(t)x'_{j,i-1}(t) + P_3(t)x_{j,i-1}(t)$$

remains positive, which is a sufficient condition for the non-oscillation of the solutions.

4. Stability and Periodicity

For a periodic system with period ω, the solutions at $t + k\omega$ can be expressed via the values in the initial period $[0, \omega]$. Specifically, for $k = 0, 1, 2, \ldots$:

$$u(k\omega + t) = u(k\omega) + u'(k\omega)t + u(t)$$

The characteristic equation for the system is given by $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$. As shown in [**?**], for stability, the coefficients must satisfy specific bounds related to the trace of the monodromy matrix. We define:

$$\Gamma = -[x_1(\omega) + x_2'(\omega) + x_3''(\omega)]$$

If $-3 < \Gamma < 1$, the system exhibits stable behavior. The values of $x_{is}(\omega)$ are calculated using the integral representations:

$$x_{11}(\omega) = 1 + \int_0^\omega u''(\tau)F_{11}(\tau)[u'(\omega) - u'(\tau)]d\tau$$

By applying the mean value theorem to the integral terms and utilizing the inequalities derived in Section 2, we can provide upper bounds for the solutions $x_{is}(t)$.

5. Numerical Example

Consider the third-order equation:

$$x''' + \mu \sin^2 tx'' + (\lambda^2 + \delta \cos 2t)x' + \eta \sin^3 2tx = 0$$

where $\mu, \gamma > 0$. For small values of the parameters $\mu, \lambda, \delta, \eta$, we can verify the stability conditions. Using the derived formulas:

$$u''(t) = \exp(-\mu \sin^2 t), \quad u'(t) = \int_0^t \exp(-\mu \sin^2 \tau)d\tau$$

The stability region in the parameter space is determined by the condition $\Gamma(\mu, \lambda, \delta, \eta) < 1$. Numerical results indicate that for $\lambda^2 > |\delta|$, the system remains stable provided the damping coefficient μ is sufficiently large to satisfy the integral inequalities.

References

1. Starzhinskii, V. M., *Bulletin of Moscow State University*, Mathematics and Mechanics Series, No. 6, pp. 818–825, 1959. (Received May 28, 1966).

Note: Figure translations are in progress. See original paper for figures.