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Abstract

Full Text

MATHEMATICS

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ON A CERTAIN PROPERTY OF BLASCHKE FUNCTIONS

1°. In the author’ s paper (1) the class $A_\alpha(-1 < \alpha < +\infty)$ of functions $f(z)$, analytic in the disk $|z| < 1$, was defined and studied, for which the integrals

$$m_\alpha(r; f) \equiv m_\alpha(r; f(z)) = \frac{r^{-\alpha}}{2\pi} \int_0^{2\pi} D_{(+)}^{-\alpha} \log|f(re^{i\theta})|\, d\theta \tag{1}$$

are bounded as $r \to 1-0$. Here $D^{-\alpha}$ $(-1 \leq \alpha < +\infty)$ is the operator of integration (for $0 < \alpha < +\infty$) or differentiation (for $-1 < \alpha < 0$) in the Riemann-Liouville sense with initial point at zero, and

$$D_{(+)}^{-\alpha}\varphi(r) = \max\{D^{-\alpha}\varphi(r), 0\}, \qquad D^{-\alpha}\varphi(r)\big|_{\alpha=0} = \varphi(r). \tag{2}$$

The classes $A_\alpha(-1 < \alpha < +\infty)$ expand monotonically as the parameter α increases, and therefore, in particular, the strict inclusion

$$A_\alpha \subset A_0 \qquad (-1 < \alpha < 0), \tag{3}$$

holds, where, by virtue of (2), $A_0 = A$ is the well-known class of A. Ostrovskii.

It is also known that if a function $f(z) \not\equiv 0$ belongs to the class A_α and $\{z_k\}_1^\infty$ $(0 < |z_k| \leq |z_{k+1}| < 1)$ is the sequence of its zeros, different from $z = 0$, then the quantity

$$\sigma_\alpha\{z_k\} = \sum_{k=1}^{\infty}(1-|z_k|)^{1+\alpha} \tag{4}$$

is always finite. On the other hand, if $\sigma_0\{z_k\} < +\infty$, then the Blaschke function

$$B(z) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \overline{z}_k z} \frac{|z_k|}{z_k} \qquad (|z| < 1) \tag{5}$$

with zeros at the points of the sequence $\{z_k\}_1^\infty$ is analytic and bounded in modulus by one in the disk $|z| < 1$, and therefore, obviously, belongs to the class $A = A_0$.

2°. In the present note an answer is given to the question: when can one assert that a function $B(z) \in A_0$ belongs to a given class A_α $(-1 < \alpha < 0)$.

Theorem 1. In order that the function $B(z)$ belong to the class $A_\alpha(-1 < \alpha < 0)$, it is necessary and sufficient that the condition $\sigma_\alpha\{z_k\} < +\infty$ be fulfilled.

Proof. a) Since, by virtue of the property of the class A_α, the necessity is obvious, we need to establish the sufficiency of the condition $\sigma_\alpha\{z_k\} < +\infty$ of the theorem. b) Let us denote

$$A_0(z;r) = \frac{\zeta - z}{1-\bar{\zeta}z}\,\frac{|\zeta|}{\zeta} \qquad (0 < |\zeta| < 1) \tag{6}$$

and prove the formula

$$\Omega_\alpha(re^{i\varphi};\zeta) \equiv r^{-\alpha}D^{-\alpha}\log|A_0(re^{i\varphi};\zeta)| = -\frac{1}{\Gamma(1+\alpha)}\log\frac{1}{|\zeta|} +$$

$$+\frac{1-|\zeta|^2}{\Gamma(1-\alpha)}\int_0^1 \frac{x(1+|\zeta|^2) - |\zeta|r^{-1}(1+r^2x^2)\cos(\varphi - \arg\zeta)}{|x-\zeta/re^{i\varphi}|^2\,|1-x\bar{\zeta}re^{i\varphi}|^2}(1-x)^\alpha\,dx. \tag{7}$$

For this purpose, first of all, we note that the formula

$$r^{-\alpha}D^{-\alpha}\log\left|1-\frac{re^{i\varphi}}{\zeta}\right| = \operatorname{Re}\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(1-x)^\alpha}{x-\zeta/re^{i\varphi}}\,dx \tag{8}$$

is valid for all r $(0 \le r \le 1)$, φ $(0 \le \varphi \le 2\pi)$, and ζ $(0 < |\zeta| < 1)$, where in the case $\varphi = \arg\zeta$, $|\zeta| \le r \le 1$, the integral should be understood in the sense of the principal value (see (1), Lemma 9.3). Further, since, by definition, for $-1 < \alpha < 0$

$$r^{-\alpha}D^{-\alpha}\log|1-\bar{\zeta}re^{i\varphi}| = r^{-\alpha}\frac{d}{dr}D^{-(1+\alpha)}\log|1-\bar{\zeta}re^{i\varphi}|,$$

and, as is easy to see,

$$D^{-(1+\alpha)}\log|1-\bar{\zeta}re^{i\varphi}| = -\operatorname{Re}\frac{1}{\Gamma(2+\alpha)}\int_0^r\frac{(r-t)^{\alpha+1}}{1-\bar{\zeta}te^{i\varphi}}\,\bar{\zeta}e^{i\varphi}\,dt,$$

we shall have

$$r^{-\alpha}D^{-\alpha}\log|1-\bar{\zeta}re^{i\varphi}| = -\operatorname{Re}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{(1-x)^\alpha}{1-\bar{\zeta}xre^{i\varphi}}\,\bar{\zeta}re^{i\varphi}\,dx. \qquad (9)$$

Finally, taking into account the elementary formula

$$r^{-\alpha}D^{-\alpha}\{\log|\zeta|\} = -\frac{1}{\Gamma(1+\alpha)}\log\frac{1}{|\zeta|},$$

from (6), (8), and (9) we obtain

$$\Omega_\alpha(re^{i\varphi};\zeta) = -\frac{1}{\Gamma(1+\alpha)}\log\frac{1}{|\zeta|}+$$

$$+\frac{1}{\Gamma(1+\alpha)}\operatorname{Re}\int_0^1\left\{\frac{1}{x-\zeta/re^{i\varphi}}+\frac{\bar{\zeta}re^{i\varphi}}{1-x\bar{\zeta}re^{i\varphi}}\right\}(1-x)^\alpha\,dx,$$

whence formula (7) follows immediately. Introduce the following notation:

$$I_\alpha^{(1)}(re^{i\varphi};\zeta) = 2\int_0^1 \frac{|\zeta|r^{-1}(1+r^2x^2)\sin^2((\varphi-\arg\zeta)/2)}{|x-\zeta/re^{i\varphi}|^2\,|1-x\bar{\zeta}re^{i\varphi}|^2}(1-x)^\alpha\,dx, \qquad (10)$$

$$I_\alpha^{(2)}(re^{i\varphi};\zeta) = \int_0^1 \frac{x(1+|\zeta|^2)-|\zeta|r^{-1}(1+r^2x^2)}{|x-\zeta/re^{i\varphi}|^2\,|1-x\bar{\zeta}re^{i\varphi}|^2}(1-x)^\alpha\,dx, \qquad (11)$$

with the aid of which we write formula (7) in the form

$$\Omega_\alpha(re^{i\varphi};\zeta) = -\frac{1}{\Gamma(1+\alpha)}\log\frac{1}{|\zeta|}+\frac{(1-|\zeta|)^2}{\Gamma(1+\alpha)}\left\{I_\alpha^{(1)}(re^{i\varphi};\zeta)+I_\alpha^{(2)}(re^{i\varphi};\zeta)\right\}. \qquad (7')$$

b) We pass to estimating the integrals $I_\alpha^{(k)}(re^{i\varphi};\zeta)$ $(k=1,2)$, assuming that $0\le r<1$, $0<|\zeta|<1$. From (10), for $\varphi\ne\arg\zeta$, we obtain

$$\left|I_\alpha^{(1)}(re^{i\varphi};\zeta)\right| \le \frac{|\zeta|(\varphi-\arg\zeta)^2}{r\delta^2(\varphi;\zeta)}\int_0^1\frac{(1-x)^\alpha}{|1-x\bar{\zeta}re^{i\varphi}|^2}\,dx,$$

where

$$\delta(\varphi;\zeta) = \min_{0\le x\le 1}\left|x-\frac{\zeta}{re^{i\varphi}}\right| \ge \begin{cases}(|\zeta|/r)|\sin(\varphi-\arg\zeta)|, & |\varphi-\arg\zeta|<\pi/2,\\ |\zeta|/r, & \pi/2\le|\varphi-\arg\zeta|\le\pi.\end{cases}$$

Therefore, in general, for any r $(0 < r < 1)$, ζ $(0 < |\zeta| < 1)$, and φ $(0 \le \varphi \le 2\pi)$ the estimate

$$\left|I_\alpha^{(1)}(re^{i\varphi};\zeta)\right| \le \frac{\pi^2}{|\zeta|}\int_0^1 \frac{(1-x)^\alpha}{|1-x\bar\zeta re^{i\varphi}|^2}\,dx \tag{12}$$

holds.

Let us now note that

$$x(1+|\zeta|^2) - |\zeta|r^{-1}(1+r^2x^2) = (1-x|\zeta|r)(x-|\zeta|/r),$$

and also that

$$|1-x\bar\zeta re^{i\varphi}| \ge 1-x|\zeta|r \ge 1-x|\zeta| \qquad (0\le x\le 1,\ 0\le r\le 1).$$

Then from (11) it follows that

$$\left|I_\alpha^{(2)}(re^{i\varphi};\zeta)\right| \le \int_0^1 \frac{|x-|\zeta|/r|}{|x-(\zeta/r)e^{-i\varphi}|^2(1-|\zeta|x)}(1-x)^\alpha\,dx \tag{13}$$

for any $\varphi \ne \arg\zeta$. We next estimate the integrals

$$U_k(r;\zeta) \equiv \frac{1}{2\pi}\int_0^{2\pi}\left|I_\alpha^{(k)}(re^{i\varphi};\zeta)\right|d\varphi,$$

using Poisson’ s integral formula

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\omega^2}{|1-\omega e^{i(\varphi-\psi)}|^2}\,d\varphi = 1 \qquad (0\le\omega<1,\ 0\le\psi\le 2\pi). \tag{14}$$

First, integrating inequality (12) with respect to φ, by virtue of (14), we obtain

$$U_1(r;\zeta) \le \frac{\pi^2}{|\zeta|}\int_0^1\frac{(1-x)^\alpha}{1-x^2|\zeta|^2r^2}\,dx \le \frac{\pi^2}{|\zeta|}\int_0^1\frac{(1-x)^\alpha}{1-|\zeta|x}\,dx \qquad (0<r<1), \tag{15}$$

where the interchange of the order of integration is, of course, permissible.

Proceeding analogously with inequality (13), and observing that, according to (14),

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{|x-(\zeta/r)e^{-i\varphi}|^2}=\frac{1}{|x^2-|\zeta|^2/r^2|}\leq\frac{1}{|\zeta|}\frac{1}{|x-|\zeta|/r|},\qquad x\neq|\zeta|/r,$$

we obtain the estimate

$$U_2(r;\zeta)\leq\frac{1}{|\zeta|}\int_0^1\frac{(1-x)^\alpha}{1-|\zeta|x}\,dx, \tag{16}$$

where here too the interchange of the order of integration is permissible, for example by Fubini' s theorem, since in the end on the right we obtain a finite quantity.

From identity (7)–(7′), in view of estimates (15) and (16), we arrive at the inequality

$$m_\alpha(r;A_0(z;\zeta))=\frac{1}{2\pi}\int_0^{2\pi}\left|\Omega_\alpha(re^{i\varphi};\zeta)\right|_{(+)}d\varphi\leq$$

$$\leq\frac{1}{2\pi}\int_0^{2\pi}\left|\Omega_\alpha(re^{i\varphi};\zeta)\right|d\varphi\leq\frac{1}{\Gamma(1+\alpha)}\log\frac{1}{|\zeta|}+$$

$$+\frac{2(1+\pi^2)}{\Gamma(1+\alpha)|\zeta|(1-|\zeta|)}\int_0^1\frac{(1-x)^\alpha}{1-|\zeta|x}\,dx\qquad(0<r<1). \tag{17}$$

Let us now estimate the integrals

$$V_\alpha(\zeta)=\int_0^1\frac{(1-x)^\alpha}{1-|\zeta|x}\,dx=\sum_{k=0}^\infty\frac{\Gamma(1+\alpha)\Gamma(1+k)}{\Gamma(2+\alpha+k)}|\zeta|^k\qquad(0<|\zeta|<1). \tag{18}$$

Since, by Stirling' s formula, as $k\to\infty$,

$$\frac{\Gamma^2(1+k)}{\Gamma(2+\alpha+k)\Gamma(-\alpha+k)}\sim1,$$

then there exists a constant $C_\alpha>0$, independent of $k\geqslant0$, such that

$$\frac{\Gamma(1+\alpha)\Gamma(1+k)}{\Gamma(2+\alpha+k)}\leqslant C_\alpha\frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)\Gamma(1+k)}\qquad(k=0,1,2,...).$$

Hence, and from (18), it follows that

$$V_\alpha(\xi) \leqslant C_\alpha \sum_{k=0}^{\infty} \frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)\Gamma(1+k)} |\xi|^k = C_\alpha (1-|\xi|)^\alpha \qquad (0<|\xi|<1). \tag{19}$$

From (17) and (19) it follows that, for $0<r<1$, $|z_1| \leqslant |\zeta| < 1$,

$$m_\alpha(r; A_0(z;\zeta)) \leqslant C(\alpha; |z_1|)(1-|\zeta|)^{1+\alpha}, \tag{20}$$

where $C(\alpha; |z_1|)$ does not depend on $|\zeta|$.

Finally, from (20) we obtain the inequality

$$m_\alpha(r; B) \leqslant \sum_{k=1}^{\infty} m_\alpha(r; A_0(z; z_k)) \leqslant C(\alpha; |z_1|) \sum_{k=1}^{\infty} (1-|z_k|)^{1+\alpha} \qquad (0<r<1),$$

i.e. our assertion $B(z) \in A_\alpha$.

Let us note that Blaschke functions for which $\sigma_\alpha\{z_k\} < +\infty$ $(-1<\alpha<0)$ were first considered by Frostman [(2)], who established for such functions a subtle characteristic of the exceptional set $E_\alpha \subset [0, 2\pi]$ of measure zero at the points of which the limit $|B(re^{i\theta})|$ as $r \to 1-0$ may fail to exist.

3°. Earlier we constructed [(1)] another important example of a function of the class A_α $(-1<\alpha<0)$, vanishing on the sequence $\{z_k\}_1^\infty$, $\sigma_\alpha\{z_k\} < +\infty$. This function is constructed in the form of the product

$$B_\alpha(z; z_k) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{-W_\alpha(z; z_k)}, \tag{21}$$

where

$$W_\alpha(z;\zeta) = \frac{1}{2\pi} \int_0^{2\pi} S_\alpha(e^{-i\theta} z) V_\alpha(e^{i\theta}; \zeta)\, d\theta, \tag{22}$$

$$S_\alpha(z) = \frac{1}{\Gamma(1+\alpha)} \left\{ \frac{2}{(1-z)^{1+\alpha}} - 1 \right\}, \qquad V_\alpha(re^{i\theta}; \xi) = r^{-\alpha} D^{-\alpha} \log \left| 1 - \frac{re^{i\theta}}{\xi} \right|. \tag{23}$$

Theorem 2. *Under the condition $\sigma_\alpha\{z_k\} < +\infty$ $(-1<\alpha<0)$, the representation*

$$B_\alpha(z; z_k) = B(z) \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} S_\alpha(e^{-i\theta}z)\, d\omega(\theta)\right\} \tag{24}$$

holds, where $\omega(\theta)$ *is a certain function of bounded variation on* $[0; 2\pi]$.

Proof. According to Theorem 1, $B(z) \in A_\alpha$, and since we also have $B_\alpha(z; z_k) \in A_\alpha$, the function analytic in the disk $|z| < 1$,

$$B_\alpha(z; z_k)B^{-1}(z) \neq 0 \qquad (|z| < 1),$$

will also belong to the class A_α. But then representation (24) follows directly from the general representation of functions of the class A_α (see [(1)], Theorem 9.13).

Remark. The property $\operatorname{Re} S_\alpha(z) \geqslant 0$ $(|z| < 1)$ for $-1 < \alpha < 0$ of the kernel S_α, as well as the recently established inequality [(3)]

$$|B_\alpha(z; z_k)| \leqslant |B(z)| \qquad (|z| < 1),$$

give some grounds for conjecturing that Theorem 2 remains valid in a stronger form, namely with the additional assertion that in representation (24) the function $\omega(\theta)$ is nonincreasing.

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Note: Figure translations are in progress. See original paper for figures.