ON THE BRANCHING OF PERIODIC SOLUTIONS OF AUTONOMOUS SYSTEMS AND DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

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ON THE BRANCHING OF PERIODIC SOLUTIONS OF AUTONOMOUS SYSTEMS AND DIFFERENTIAL EQUATIONS IN BANACH SPACES

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1. In paper ([1]) we proposed a method for finding the number of all solutions and the form of each solution of Poincaré' s problem on periodic solutions of nonautonomous systems with analytic right-hand sides. It turns out that analogous considerations are applicable to the solution of the problem on the branching of periodic solutions of autonomous systems with analytic right-hand sides and nonautonomous differential equations in Banach spaces.

2. Consider the system

$$dx/dt = f(x) + \lambda g(x, \lambda), \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)$ is a vector in the real space E_n; λ is a small real parameter; $f(x)$ and $g(x, \lambda)$ are analytic functions of x and λ with values in E_n.

Let $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$ be an ω-periodic solution of the generating system

$$dy/dt = f(y). \tag{2}$$

The problem is to find all $T(\lambda)$-periodic solutions $\psi(t, \lambda)$ of system (1), satisfying the conditions $\psi(t_0, 0) = \varphi(t)$ and $T(0) = \omega$. Here the continuity of $\psi(t, \lambda)$ and $T(\lambda)$ is assumed. Denote by $x(t, \alpha, \lambda)$ the solution of the initial-value problem $x(0, \alpha, \lambda) = \varphi(0) + \alpha$ for system (1). Without loss of generality (see, for example, ([3])) one may assume that $\varphi_1(0) = 0$ and $\alpha_1 = 0$. In order that the solution $x(t, \alpha, \lambda)$ be periodic with period $T = \omega + \tau(\lambda)$, it is necessary and sufficient that

$$x_1(\omega + \tau, \alpha, \lambda) = 0, \qquad x_i(\omega + \tau, \alpha, \lambda) - \varphi_i(0) - \alpha_i = 0 \quad (i = 2, 3, \dots, n), \tag{3}$$

where the left-hand sides are analytic functions at the origin of coordinates. We reduce the left-hand sides of each equation of system (3) by the highest admissible power of λ. We then obtain

$$\Psi_i(\tau, \alpha_2, \dots, \alpha_n, \lambda) = 0 \quad (i = 1, 2, \dots, n). \tag{4}$$

We shall assume that $\Psi_i(0, 0, \dots, 0) = 0$ for all i, since otherwise the problem posed has no solutions. Denote by J the Jacobi matrix at zero of $\Psi_1, \dots, \Psi_n$ with respect to $\tau, \alpha_2, \dots, \alpha_n$, and let r be the defect of this matrix. Eliminating $(n - r)$ unknowns from system (4), we obtain

$$\Phi_i(\xi_1, \dots, \xi_r, \lambda) = 0 \quad (i = 1, 2, \dots, r), \tag{5}$$

where $\xi_1, \dots, \xi_r$ are the remaining unknowns from the collection $\tau, \alpha_2, \dots, \alpha_n$. The functions Φ_i are analytic at the origin of coordinates, and moreover $\operatorname{ord} \Phi_i(\xi_1, \dots, \xi_r, 0) \geq 2$, $\operatorname{ord} \Phi_i(0, \dots, 0, \lambda) \geq 1$, so that system (5) is the branching equation of the problem under consideration.

Thus, the solution of the problem on the branching of periodic solutions of system (1) is reduced to finding all small real solutions of the branching equation (5).

Using the method developed by us in paper ([2]), we arrive at various propositions on the number and form of all solutions of the problem posed. We give one such proposition. Let $d_i(\xi_i^{(i)}, \dots, \xi_r^{(i)}, \lambda)$ be the distinguished polynomials with respect to $\xi_i^{(i)}$ (or polynomials of degree zero), obtained for system (5) by the procedure indicated in ([2]).

Theorem 1. *Let $r \geqslant 1$. Then, if $d_k \sim 1$ $(k = 1, 2, \dots, r - 1)$ and d_r is not associated with unity, then the problem under consideration has a finite number of solutions, and each of them is representable, for small $|\lambda|$, in the form of a convergent series in powers of $\lambda^{1/p}$, where p is some natural number. All these solutions are periodic with period*

$$T = \omega + \sum_{k=1}^{\infty} a_k \lambda^{k/p},$$

and this series also converges for small $|\lambda|$.

Let us note that for even p the corresponding real solution is defined only for $\lambda \geqslant 0$. To find, in this case, the solution for $\lambda \leqslant 0$, one must replace λ by $-\lambda$ in system (1) and repeat all calculations. This remark also applies to Sec. 3.

It is advisable to combine the method proposed here with the method of undetermined coefficients. Namely, under the hypotheses of Theorem 1, we first obtain information on the form of the series for α_i $(i = 2, \dots, n)$ and τ, and then, by means of the substitution

$$t = \theta\left(1 + \frac{1}{\omega}\tau(\lambda)\right)$$

in system (1) (see, for example, ([4,5])), we obtain

$$\frac{dx}{d\theta} = f(x) + \frac{\tau(\lambda)}{\omega}f(x) + \lambda\left(1 + \frac{\tau(\lambda)}{\omega}\right)g(x,\lambda). \tag{1'}$$

We seek the solution of system $(1')$ in the form of a series in powers of $\lambda^{1/p}$ with undetermined ω-periodic coefficients.

3. Consider the equation, important in applications,

$$d^2x/dt^2 + k^2x = \lambda f(x, \dot{x}, \lambda), \tag{6}$$

where $f(x, \dot{x}, \lambda)$ is an analytic function in some domain of variation of the arguments. The solution of the initial-value problem $y(0) = A_0$, $\dot{y}(0) = 0$ for the generating equation $\ddot{y} + k^2y = 0$ will be

$$y = A_0 \cos kt.$$

Without loss of generality ([4]), in the case of small $|\lambda|$ one may take the following initial conditions for equation (6):

$$x(0) = A_0 + \alpha, \qquad \dot{x}(0) = 0. \tag{7}$$

Let $x(t, \alpha, \lambda)$ be the solution of problem (6)–(7). Then, in order that this solution be periodic with period $T = 2\pi/k + \tau(\lambda)$, it is necessary and sufficient that

$$x(2\pi/k + \tau, \alpha, \lambda) - A_0 - \alpha = 0, \qquad \dot{x}(2\pi/k + \tau, \alpha, \lambda) = 0, \tag{8}$$

where the left-hand sides are analytic functions of τ, α, λ at the point $\tau = \alpha = \lambda = 0$. Reducing the left-hand sides of equalities (8) by the highest admissible powers of λ, we obtain the system

$$\Psi_i(\tau, \alpha, \lambda) = 0 \qquad (i = 1, 2). \tag{9}$$

As before, it is assumed that $\Psi_i(0, 0, 0) = 0$. Let J be the Jacobi matrix at zero of Ψ_1, Ψ_2 with respect to τ and α.

In the case when J is a nonsingular matrix, as is known [4], the problem has a unique solution. The case when $\operatorname{rang} J = 1$ was studied in [5]. We investigate the case when $\operatorname{rang} J = 0$. Such a situation is possible for $A_0 = 0$. In this case system (9) is the branching equation of the problem under consideration. At the same time we exclude the trivial case when $\Psi_i \equiv 0$ for $i = 1, 2$.

By means of a nonsingular linear change of variables

$$(\tau, \alpha) = P(\xi_1, \xi_2)$$

and the preparatory Weierstrass theorem, system (9) is reduced to an equivalent system (with respect to small solutions)

$$G_i(\xi_1, \xi_2, \lambda) = 0 \quad (i = 1, 2), \tag{10}$$

where G_i are distinguished polynomials with respect to ξ_1.

Form the resultant, with respect to ξ_1, of the polynomials G_1 and G_2, set it equal to zero, and reduce this equality by the highest admissible power of λ. We obtain

$$R(\xi_2, \lambda) = 0. \tag{11}$$

If $R(\xi_2, \lambda) \not\equiv 0$ and $R(0, 0) = 0$, then by the Newton diagram method we find all small real solutions of equation (11). These solutions are represented in the form of convergent series in powers of $\lambda^{1/q}$, where q is some natural number. Setting $\lambda^{1/q} = \mu$ and substituting each small real solution of equation (11) into (10), we arrive at the equation

$$d(\xi_1, \mu) = 0, \tag{12}$$

where d is the greatest common divisor of the distinguished polynomials

$$G_i\left(\xi_1, \sum_{k=1}^{\infty} a_k \mu^k, \mu^q\right).$$

From (12), by the Newton diagram method, we find the real solutions ξ_1 as functions of μ. Hence it follows

Theorem 2. *If $R(\xi_2, \lambda) \not\equiv 0$ and $R(0, 0) = 0$, then the stated problem has a finite number of solutions, and each of them, as well as $\tau(\lambda)$, is representable for small $|\lambda|$ in the form of convergent series in powers of $\lambda^{1/p}$, where p is some natural number. If $R(0, 0) \neq 0$, then the problem under consideration certainly has no solutions.*

We note that, in contrast to the complex case, under the conditions of Theorems 1 and 2 the problem may also have no real solutions. We also note that some general considerations on autonomous systems are contained in [8].

4. The problem of branching of periodic solutions also arises in the study of differential equations in Banach spaces. Let E be a Banach space (complex or real), U the real axis, and Λ the complex plane (or the real axis). Further, let $f(t,x)$ and $g(t,x,\lambda)$ be functions with values in E, defined in some domain D of variation of the arguments $t \in U$, $x \in E$, $\lambda \in \Lambda$. We shall assume that the functions $f(t,x)$ and $g(t,x,\lambda)$ are continuous in the totality of their arguments in their domain of definition, bounded, ω-periodic in t, and analytic in the totality of x and λ. Consider the equation

$$dx/dt = f(t,x) + \lambda g(t,x,\lambda). \tag{13}$$

It follows from the listed conditions that, for every fixed λ, the Cauchy problem for equation (13) has a unique solution, continuous in t.

Let the generating equation

$$dy/dt = f(t,y) \tag{14}$$

have an ω-periodic solution $\varphi(t)$. The problem is posed of finding all ω-periodic solutions of equation (13), continuous in t and λ and tending to $\varphi(t)$ as $\lambda = 0$.

Following Poincaré’ s idea, let us consider, for equation (13), the initial-value problem

$$x(0,\alpha,\lambda) = \varphi(0) + \alpha, \tag{15}$$

where $x(t,\alpha,\lambda)$ is the solution of this initial-value problem. It turns out that the solution $x(t,\alpha,\lambda)$ can be written in the form

$$x(t,\alpha,\lambda) = \varphi(t) + \chi(t,\alpha,\lambda), \tag{16}$$

where $\chi(t,\alpha,\lambda)$ is an (F)-power series (6)

$$\chi(t,\alpha,\lambda) = \sum_{m+n\geq 1} a_{mn}(t)\alpha^m\lambda^n, \tag{17}$$

convergent for $\|\alpha\| \leq \varepsilon$ and $|\lambda| \leq \lambda_0$; moreover the m-linear operators $a_{mn}(t)$ are determined uniquely by substituting (17) into (13) and solving the resulting recurrent system of linear differential equations for them with the initial conditions (15), which take the form

$$a_{mn}(0) = \begin{cases} I & \text{for } m = 1,\ n = 0, \\ \theta & \text{for all other values of } m \text{ and } n. \end{cases}$$

Here I is the identity operator from E into E, and θ is the zero element of the corresponding space.

In order that the solution (16) be ω-periodic, it is necessary and sufficient that

$$F(\alpha, \lambda) \equiv \chi(\omega, \alpha, \lambda) - \alpha = 0. \tag{18}$$

Putting $B = I - a_{10}(\omega)$ and $F_{mn} = a_{mn}(\omega)$, we obtain

$$B\alpha = F_{01}\lambda + \sum_{m+n\geq 2} F_{mn}\alpha^m\lambda^n. \tag{19}$$

The solution of the problem posed is therefore reduced to finding all small solutions of equation (19).

Let B be a normally solvable operator of index zero, and let r be the dimension of the null subspace of this operator. These conditions are satisfied if $a_{10}(\omega)$ is a completely continuous operator for which 1 is an r-fold eigenvalue. To find the small solutions of equation (19), we derive a branching equation, which in the present case takes the form

$$\Phi_i(\xi_1, \xi_2, \dots, \xi_r, \lambda) = 0 \quad (i = 1, 2, \dots, r), \tag{20}$$

with the same properties as system (5). Using the results of work [2], we arrive at various propositions on the number and form of the solutions of the problem posed concerning the branching of solutions of equation (13).

Let us note that various classes of integro-differential equations, as well as countable systems of ordinary differential and integro-differential equations, can be reduced to equation (13). Let us also note that, in the absence of branching, the Poincaré problem for countable systems of ordinary differential equations was studied in work [7].

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Note: Figure translations are in progress. See original paper for figures.