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The solution of many applied problems leads to the solution of the transcendental equation

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Abstract

Full Text

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ON THE THEORY OF ITERATIVE PROCESSES

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The solution of many applied problems leads to the solution of the transcendental equation

$$f(x) = 0. \tag{1}$$

There are many methods for solving it, the most widespread being iterative ones. To justify the convergence of an iterative process for $f(x)$, boundedness of the derived numbers in a neighborhood of its zeros is required. Generally speaking, the convergence of the process depends on the initial approximations. In paper (1), iterative processes were constructed which, under one-sided boundedness of the derived numbers of $f(x)$, converge to the minimal zero of $f(x)$ on (x_0, b) or to the maximal zero of $f(x)$ on (a, x_0) (independently of the initial approximations).

In the present paper iterative processes are constructed and investigated which converge to the minimal zero of $f(x)$ on (x_0, b) or to the maximal zero of $f(x)$ on (a, x_0) (independently of the initial approximations). Here only continuity of $f(x)$ on (a, b) is required.

For simplicity in the formulation of the theorem, we adopt the following notation. Denote by $\mathcal{K}$ the set of all functions in $C(-\infty, \infty)$ that are strictly increasing and tend to $+\infty$ or to $-\infty$ as $x \to +\infty$ or $x \to -\infty$, respectively. Denote by η_f the greatest lower bound of the set of zeros of $f(x)$ in (x_0, b), and by ξ_f the least upper bound of the set of zeros of $f(x)$ on (a, x_0). Denote by $\mathcal{E}$ the set of those $x \in (a, b)$ for which $f(x) - qf(x_0) < 0$, where $0 < q < 1$ is a certain fixed number.

Let us consider the function

$$M(q, g, x_0, x) = [f(x) - qf(x_0)]/[g(x) - g(x_0)].$$

It is not difficult to see that $M(q, g, x_0, x)$, as a function of the variable x, is continuous on (a, x_0) and (x_0, b);

$$\lim_{x \to x_0 - 0} M(q, g, x_0, x) = +\infty, \qquad \lim_{x \to x_0 + 0} M(q, g, x_0, x) = -\infty.$$

If $\mathcal{E} \cap (a,b)$ is empty, then equation (1) has no solutions in (a,b). For computing the exact lower and exact upper bounds of the function $M(q,g,x_0,x)$ with respect to x, it is useful to establish conditions under which the function $M(q,g,x_0,x)$ is monotone with respect to x.

Theorem 1. *If $f'(x)$ and $g'(x)$ exist on (x_0,b), the derivative $g'(x) > 0$ is finite, and*

$$\begin{vmatrix} f'(x) & f'(\xi) \\ g'(x) & g'(\xi) \end{vmatrix} < 0,$$

when $x_0 < \xi \le x < b$, then $M(q,g,x_0,x)$ decreases with respect to x on (x_0,b).

One can establish conditions under which the function $M(q,g,x_0,x)$ increases with respect to x on (x_0,b).

It is not difficult to prove that the function $M(q,g,x_0,x)$ is bounded below on (x_0,b) and bounded above on (a,x_0). With the aid of $M(q,g,x_0,x)$ we construct the function

$$P_+(q,g,x_0) = \inf_{x_0 < x \le b} M(1,g,x_0,x).$$

It is clear that: 1) $P_+(q,g,x_0)$ exists and is finite; 2) if $\mathcal{E} \cap (x_0,b)$ is nonempty, then $P_+(q,g,x_0) < 0$; 3) $P_+(q,ag,x_0)$ is monotone with respect to a and decreases with respect to q.

Lemma 1. If $f'(x)$ and $g'(x)$ exist, with $g'(x) > 0$ and finite on (x_0,b), then

$$P_+(q,g,x_0) \ge \inf_{x_0 \le x \le b} \frac{f'(x)}{g'(x)} + (1-q)\frac{f(x_0)}{g(b)-g(x_0)}.$$

Lemma 2. If $\mathcal{E} \cap (x_0,b)$ is empty, then

$$P_+(q,g,x_0) \ge -\frac{\max_{x_0 \le x \le b} |f(x)| + qf(x_0)}{g(x_0+\delta)-g(x_0)},$$

where $\delta > 0$ is such that $f(x) - qf(x_0) \ge 0$ on $(x_0, x_0+\delta)$.

Obviously, such a δ exists, since $f(x) \in C(a,b)$, $0 < q < 1$, and, moreover, $f(x_0) > 0$, where $x_0 \in (a,b)$.

Lemma 3. If on $[\alpha,\beta]$ $f(x) > 0$, where $[\alpha,\beta] \subset (a,b)$, then $P_+(q,g,x_0)$ is bounded below with respect to $x_0 \in [\alpha,\beta]$.

In terms of $P_+(q,g,x_0)$ one can prove existence theorems for solutions of equation (1).

Theorem 2. If

$$P_+(q, g, x_0) \le qf(x_0)/[g(x_0 + \delta) - g(x_0)]$$

(where $\delta > 0$ is such that $f(x) - qf(x_0) \ge 0$ on $(x_0, x_0 + \delta)$), then equation (1) has at least one solution on (x_0, b).

Theorem 3. If there exists a number $\xi \in (x_0, b)$ such that on (x_0, ξ) $f'(x)$ and $g'(x)$ exist and the inequality

$$\inf_{x_0 \le x \le \xi} \frac{f'(x)}{g'(x)}[g(x) - g(\xi)] \ge f(x_0),$$

holds, then equation (1) has at least one solution on (x_0, ξ).

With the aid of the function $P_+(q, g, x_0)$ one can construct an iterative process that converges to η_f.

Theorem 4. In order that the sequence obtained in the process

$$x_{n+1} = g^{-1}\left[g(x_n) - \frac{qf(x_n)}{P_+(q, g, x_n)}\right], \tag{A}$$

converge to η_f, it is necessary and sufficient that: 1) $P_+(q, g, x_n) < 0$, if $f(x_n) \ne 0$; 2) all elements of the sequence x_n do not exceed the number b.

Here g^{-1} denotes the function inverse to g, whose existence follows from the definition of the set $\mathcal{K}$.

Remark. If equation (1) has at least one solution on (x_0, b), then all the conditions of Theorem 4 are fulfilled, and the sequence $\{x_n\}$ obtained in the process (A) converges to η_f.

It is not difficult to show that if in the process (A) one replaces $g(x)$ by $mg(x)$, $m > 0$, then the same sequence $\{x_n\}$ is obtained.

One can construct the process

$$x_{n+1} = g^{-1}[g(x_n) - qf(x_n)/P_-(q, g, x_n)], \tag{A_0}$$

where

$$P_-(q, g, x_n) = \sup_{a \le x \le x_n} M(q, g, x_n, x).$$

For the process (A_0) the following holds.

Theorem 5. The sequence obtained in the process (A_0) converges to ξ_f if and only if: 1) $P_-(q, g, x_n) > 0$ for all x_n for which $f(x_n) \ne 0$; 2) all elements of the sequence $\{x_n\}$ exceed the number a.

Remark. If equation (1) has at least one solution on (a, x_0), then the sequence $\{x_n\}$ obtained in the process (A_0) converges to ξ_f.

With the help of the function

$$\overline{M}(q, g, x_0, x) = [|f(x)| - qf(x_0)]/[g(x) - g(x_0)]$$

one can construct

$$\overline{P}_+(q, g, x_0) = \inf_{x_0 \le x \le b} \overline{M}(q, g, x_0, x).$$

It is clear that $\overline{P}_+(q, g, x_0) \ge P_+(q, g, x_0)$. Therefore

$$\overline{P}_+(q, g, x_0) \ge -qf(x_0)/[g(x_0 + \delta) - g(x_0)],$$

where $\delta > 0$ is a number such that $f(x_0) - qf(x_0) > 0$ on $(x_0, x_0 + \delta)$.

If one constructs the process by the formula

$$x_{n+1} = g^{-1}[g(x_n) - qf(x_n)/\overline{P}_+(q, g, x_n)], \qquad (A_1)$$

then the conditions necessary and sufficient for the convergence of the sequence $\{x_n\}$, which is obtained in the process (A_1), are all the conditions of Theorem 4 with the sole difference that, in the statement of this theorem, instead of $P_+(q, g, x_n)$ one must take $\overline{P}_+(q, g, x_n)$. Suppose that $P_+(q, g, \xi)$ is bounded below relative to $\xi \in (x_0, b)$ by some number m, and equation (1) has at least one solution on (x_0, b). Then the sequence

$$x_{n+1} = g^{-1}[g(x_n) - qf(x_n)/m]$$

converges to η_f.

Analogously to the processes (A_0) and (A_1), one can construct processes that converge to ξ_f.

Remark. If equation (1) has at least one solution on (x_0, b), then $\overline{P}_+(q, g, x_0)$ and, consequently, $P_+(q, g, x_0)$ cannot be positive. Therefore the sequences obtained in the processes (A), (A_1), (A_2) are nondecreasing. An analogous remark holds for the process (A_0).

Theorem 6. Let

$$x_1 = g^{-1}[g(x_0) - qf(x_0)/P_+(q, g, x_0)],$$

$$\overline{x}_1 = g^{-1}[g(x_0) - qf(x_0)/\overline{P}_+(q, g, x_0)],$$

and, moreover, $\mathcal{E} \cap (x_0, b)$ be nonempty. Then $\overline{x}_1 \geq x_1$.

Theorem 7. Let in the processes

$$x_1 = g_1^{-1}[g_1(x_0) - qf(x_0)/m], \qquad \overline{x}_1 = g_2^{-1}[g_0(x_0) - qf(x_0)/m]$$

and 1) $P_+(q, g_i, x_0) \geq m$, $i = 1, 2$; 2) $g_1(x_0) = g_2(x_0)$, $g_1'(x) > g_2'(x)$ for $x \in (x_0, b)$. Then $\overline{x}_1 < x_1$.

The rate of convergence in the process (A_1) depends on q, g, and x_0. It can be shown that if in the process (A_1) $\mathcal{E} \cap (x_0, b)$ is nonempty and $q_1 > q_2$, then $x_1(q_1, g, x_0) > x_2(q_1, g, x_0)$ for identical initial approximations.

In studying the rate of convergence of the sequence $\{x_n\}$ obtained in the process (A), one has to impose on $f(x)$ and $g(x)$ certain additional conditions.

Theorem 8. Let: 1) $f(x)$ and $g(x)$ have derivatives on (ξ, η_f), with: a) $f'(x)/g'(x) < r_0 < 0$ for $x \in (\xi, \eta_f)$; b) $0 < g'(x_{n+1})/g'(x_n) < r$, for $x_n \in (\xi, \eta_f)$, where $x_0 < \xi < \eta_f$; 2) $r_2 \leq P_+(q, g, x) \leq r_3 < 0$; 3) $(qr_0 + r_3/r_1)/r_2 > 1$. Then, beginning with some index, the series

$$\sum_{n=1}^{\infty} (x_{n+1} - x_n)$$

will decrease no more slowly than a geometric progression.

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Note: Figure translations are in progress. See original paper for figures.