A METHOD FOR INVESTIGATING IRREGULAR WAVEGUIDES WITH IMPEDANCE BOUNDARY CONDITIONS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 538.566

MATHEMATICS

A. S. IL' INSKII, A. G. SVESHNIKOV

A METHOD FOR INVESTIGATING IRREGULAR WAVEGUIDES WITH IMPEDANCE BOUNDARY CONDITIONS

(Presented by Academician I. G. Petrovskii, 28 XI 1966)

In works (^{1}, ^{2}) a justification was given for an algorithm for the approximate solution of the problem of the propagation of electromagnetic oscillations in a broad class of irregular waveguides with a perfectly conducting lateral surface. In the present communication this algorithm is generalized to the case of propagation of electromagnetic waves in an irregular waveguide with impedance boundary conditions.

Let there be two semi-infinite regular waveguides with perfectly conducting walls, joined by a transition section with anisotropic filling and a lateral surface of complicated shape having finite, but large, conductivity, so that the Leontovich condition may be prescribed as the boundary condition on this surface. One of the normal waves of one of the regular waveguides is incident on the irregular section. It is required to determine the amplitudes of the normal waves propagating in both directions away from the irregular section.

The mathematical formulation of the problem of propagation of electromagnetic oscillations in such an irregular waveguide consists in determining a solution of the homogeneous system of Maxwell equations

$$\operatorname{rot}\mathbf{H} = -ik\overleftrightarrow{\varepsilon}\mathbf{E}; \qquad \operatorname{rot}\mathbf{E} = ik\overleftrightarrow{\mu}\mathbf{H} \tag{1}$$

in the domain D inside the irregular section of the waveguide, satisfying the conditions

$$[\mathbf{nE}]_\Sigma = w[\mathbf{n}[\mathbf{nH}]]_\Sigma, \qquad \operatorname{Re} w \geqslant 0; \tag{2}$$

$$\iint_{S_1} [\mathbf{HE}_m^{(1)}]_{x_3}\, d\tau = (2A\delta_{mm_0} - P_m)\beta_m^1; \tag{3}$$

$$\iint_{S_2} [\mathbf{H}\mathbf{E}_m^{(2)}]_{x_3}\, d\tau = -T_m \beta_m^2. \tag{4}$$

Here $\mathbf{E}_m^i$ are the transverse parts of the normal waves of the left ($i = 1$) and right ($i = 2$) regular waveguides with transverse sections S_1 and S_2, respectively. The function $A(x_3)$ determines the amplitude of the incident normal wave of number m_0 of the left regular waveguide. The functions $P(x_3)$ and $T(x_3)$ determine the reflection and transmission coefficients of the normal waves scattered by the irregular section of the waveguide.

For solving the boundary-value problem (1)–(4) for a domain with a complicated boundary Σ, just as in works ([1]-[3]), by introducing a special coordinate system we pass to a boundary-value problem for one of the simple standard domains. We shall assume that a mapping of the irregular waveguide onto a regular cylinder has been chosen. We write this mapping in general form

$$\xi_i = \xi_i(x_1, x_2, x_3), \qquad i = 1, 2, 3, \tag{5}$$

where x_i are Cartesian coordinates. We require that the transformation (5) be nonsingular and that the axis ξ_3 pass into the axis x_3 in the regular sections of the waveguide, i.e., that the regular sections of the waveguide be deformed only

in the plane of the transverse section. Let us introduce into consideration the base coordinate vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and the reciprocal coordinate vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ ([4]). To write equations (1) in the new coordinate system it is convenient to use the covariant (ε_i, h_i) coordinates of the vectors $\mathbf{E}, \mathbf{H}$. As is easily shown, for the vectors $\mathbf{E}', \mathbf{H}'$ in the orthogonal curvilinear coordinate system defined by the vectors $\mathbf{e}_{\xi_i}$, we obtain the system of equations

$$\operatorname{rot} \mathbf{H}' = ik\overset{\leftrightarrow}{\varepsilon}\, \mathbf{E}'; \qquad \operatorname{rot} \mathbf{E}' = -ik\overset{\leftrightarrow}{\mu}\, \mathbf{H}', \tag{6}$$

where the vectors $\mathbf{E}'\mathbf{H}'$ are determined by the relations

$$\mathbf{E}' = \sum_{i=1}^{3} h_{\xi_i} \varepsilon_i \mathbf{e}_{\xi_i}; \qquad \mathbf{H}' = \sum_{i=1}^{3} h_{\xi_i} h_i \mathbf{e}_{\xi_i}; \tag{7}$$

h_{ξ_i} are the Lamé coefficients of the orthogonal coordinate system $\mathbf{e}_{\xi_i}$. The tensors $\overset{\leftrightarrow}{\varepsilon}, \overset{\leftrightarrow}{\mu}$ are expressed as follows in terms of the metric coefficients of the curvilinear coordinate system (5):

$$\overset{\leftrightarrow}{\varepsilon} = \sqrt{g}\begin{vmatrix} \dfrac{h_{\xi_1}}{h_{\xi_2}h_{\xi_3}}\widehat{\varepsilon}^{11} & \dfrac{1}{h_{\xi_3}}\widehat{\varepsilon}^{12} & \dfrac{1}{h_{\xi_1}}\widehat{\varepsilon}^{13} \\ \dfrac{1}{h_{\xi_3}}\widehat{\varepsilon}^{21} & \dfrac{h_{\xi_2}}{h_{\xi_1}h_{\xi_3}}\widehat{\varepsilon}^{22} & \dfrac{1}{h_{\xi_2}}\widehat{\varepsilon}^{23} \\ \dfrac{1}{h_{\xi_1}}\widehat{\varepsilon}^{31} & \dfrac{1}{h_{\xi_2}}\widehat{\varepsilon}^{23} & \dfrac{h_{\xi_3}}{h_{\xi_2}h_{\xi_1}}\widehat{\varepsilon}^{33} \end{vmatrix}; \tag{8}$$

$$\overset{\leftrightarrow}{\mu} = \sqrt{g}\begin{vmatrix} \dfrac{h_{\xi_1}}{h_{\xi_2}h_{\xi_3}}\widehat{\mu}^{11} & \dfrac{1}{h_{\xi_3}}\widehat{\mu}^{12} & \dfrac{1}{h_{\xi_1}}\widehat{\mu}^{13} \\ \dfrac{1}{h_{\xi_3}}\widehat{\mu}^{21} & \dfrac{h_{\xi_2}}{h_{\xi_3}h_{\xi_1}}\widehat{\mu}^{22} & \dfrac{1}{h_{\xi_2}}\widehat{\mu}^{23} \\ \dfrac{1}{h_{\xi_1}}\widehat{\mu}^{21} & \dfrac{1}{h_{\xi_2}}\widehat{\mu}^{32} & \dfrac{h_{\xi_3}}{h_{\xi_1}h_{\xi_2}}\widehat{\mu}^{33} \end{vmatrix}; \tag{9}$$

$$\widehat{\mu}^{ij} = \sum_{\alpha\beta}^{3} \frac{\partial\xi^i}{\partial x^\alpha}\frac{\partial\xi^j}{\partial x^\beta}\mu^{\alpha\beta}, \qquad \widehat{\varepsilon}^{ij} = \sum_{\alpha\beta}^{3} \frac{\partial\xi^i}{\partial x^\alpha}\frac{\partial\xi^j}{\partial x^\beta}\varepsilon^{\alpha\beta}; \tag{10}$$

$\mu^{\alpha\beta}, \varepsilon^{\alpha\beta}$ are the components of the tensors $\overset{\leftrightarrow}{\varepsilon}, \overset{\leftrightarrow}{\mu}$ in the Cartesian coordinate system.

We shall assume that the lateral surface in the new coordinate system coincides with one of the coordinate surfaces

$$\xi_1 = \text{const.}$$

In this case the boundary conditions (2) on the impedance wall can be rewritten in the following form:

$$[\mathbf{e}_{\xi_1}\mathbf{E}']\big|_{\Sigma} = w^{\leftrightarrow}[\mathbf{e}_{\xi_1}[\mathbf{e}_{\xi_1}\mathbf{H}']]\big|_{\Sigma}. \tag{11}$$

The tensor $\overset{\leftrightarrow}{w}$ is expressed in terms of the metric coefficients (^{4}) of the new coordinate system as

$$\overset{\leftrightarrow}{w} = \frac{w}{(g^{11})^{1/2}}\begin{vmatrix} (g^{11}g^{22} - (g^{12})^2)\,\dfrac{h_{\xi_3}}{h_{\xi_2}^{-1}} & (g^{12}g^{13} - g^{11}g^{32}) \\ (g^{12}g^{13} - g^{11}g^{32}) & (g^{11}g^{33} - (g^{13})^2)\,\dfrac{h_{\xi_2}}{h_{\xi_3}^{-1}} \end{vmatrix}. \tag{12}$$

The conditions on the regular perfectly conducting sections of the waveguide can be rewritten in the form

$$\iint_{S_1} [\mathbf{H}'\mathbf{e}_n]_{\xi_3} \sqrt{g}\, d\sigma = -\sum_{m=1}^{\infty} \alpha^1_{nm} \beta^1_m P_m + 2\alpha^{1*}_{nm_0} \beta^1_{m_0} A; \tag{13}$$

$$\iint_{S_2} [\mathbf{H}'\mathbf{e}_n]_{\xi_3} \sqrt{g}\, d\sigma = -\sum \alpha^2_{nm} \beta^2_m T_m. \tag{14}$$

Here $\mathbf{e}_n$ are the transverse parts of the normal waves of the regular hollow cylinder onto which the irregular waveguide has been mapped; α^i_{nm} are the coefficients of the expansions of the functions $\mathbf{e}_n$ in the normal waves of the original regular waveguides. The original boundary-value problem for system (1) with conditions (2)–(4) is equivalent to the boundary-value problem for Maxwell' s system of equations (6) with conditions (11)–(14).

The basic idea in constructing an approximate solution of the given problem is the transition to a boundary-value problem for a finite system of ordinary differential equations, carried out by a method analogous to the Galerkin method.

We seek the transverse components of the approximate solution in the form

$$\mathbf{E}'^N_t = \sum_{n=1}^{N} A_n \mathbf{e}_n, \qquad \mathbf{H}'^N_t = \sum_{n=1}^{N} B_n \mathbf{h}_n. \tag{15}$$

We determine the longitudinal components from the relations

$$(\operatorname{rot} \mathbf{H}'^N_t)_{\xi_3} = -ik(\overset{\leftrightarrow}{\varepsilon} \mathbf{E}'^N)_{\xi_3}, \qquad (\operatorname{rot} \mathbf{E}'^N_t)_{\xi_3} = ik(\overset{\leftrightarrow}{\mu} \mathbf{H}'^N)_{\xi_3}. \tag{16}$$

To determine A_n and B_n, we require that, for all ξ_3, the approximate solution satisfy the integral relations

$$\iint_{S(\xi_3)} (\operatorname{rot} \mathbf{H}'^N + ik\overset{\leftrightarrow}{\varepsilon} \mathbf{E}'^N)_t \mathbf{e}_n \, d\sigma = f^1_n; \tag{17}$$

$$\iint_{S(\xi_3)} (\operatorname{rot} \mathbf{E}'^N - ik\overset{\leftrightarrow}{\mu} \mathbf{H}'^N)_t \mathbf{h}_n \, d\sigma = f^2_n; \tag{18}$$

$$f^1_n = \operatorname{Re} w^{-1} \oint_C \frac{g^{33}}{h_{\xi_2}}\, g^{22} (\overset{\leftrightarrow}{\varepsilon} \mathbf{E}^N_t)_{\xi_3} (\overset{\leftrightarrow}{\varepsilon} \mathbf{e}_n)_{\xi_3} \cdot \frac{1}{\sqrt{g^{11}}}\, dl; \tag{19}$$

$$f^2_n = \oint_C \left\{ E^N_{\xi_3} (\mathbf{h}\vec{\tau}) + \operatorname{Re} w^{-1} \left[\frac{k^2}{h_{\xi_1}} (g^{33})^2 \frac{4}{\sqrt{g^{11}}} (\operatorname{rot} \mathbf{h}_n)_{\xi_3} (\operatorname{rot} \mathbf{H}^N)_{\xi_3} \right] \right\} dl. \tag{20}$$

This form of the Galerkin method readily makes it possible to prove that the approximate solution satisfies the integral relation

$$\operatorname{Re}\sum_{m\neq m_0}\beta^1_m|P^N_m|^2+\operatorname{Re}\sum_m\beta^2_m|T^N_m|^2+\operatorname{Re}\beta^1_{m_0}\left|P^N_{m_0}-\frac{\beta^1_{m_0}}{\operatorname{Re}\beta^1_{m_0}}A\right|^2+$$

$$+k\ \operatorname{Im}\iiint_D\{\varepsilon_0|\mathbf{E}^N|^2+\mu_0|\mathbf{H}^N|^2\}\,dv=\frac{|\beta^1_{m_0}|^2}{\operatorname{Re}\beta^1_{m_0}}|A|^2, \tag{21}$$

$$\overset{\leftrightarrow}{\varepsilon}=\varepsilon_1+\varepsilon_0\overset{\leftrightarrow}{I},\qquad \overset{\leftrightarrow}{\mu}=\mu_1+\mu_0\overset{\leftrightarrow}{I}.$$

As is easy to show, this same integral relation is also satisfied by the exact solution $\mathbf{E},\mathbf{H}$, provided that the exact solution permits the application of the Lorentz lemma. The convergence of $\mathbf{E}^N\mathbf{H}^N$ to the exact solution $\mathbf{E},\mathbf{H}$

it follows that for the differences $\vec{\mathcal{E}}^{\,N}=\mathbf{E}-\mathbf{E}^N$ and $\vec{\mathcal{H}}^{\,N}=\mathbf{H}-\mathbf{H}^N$ an analogous energy relation holds (the notation is the same as in work (^{1})).

$$k\ \operatorname{Im}\iiint_D\{\varepsilon_0|\mathcal{E}^N|^2+\mu_0|\vec{\mathcal{H}}^{\,N}|^2\}\,dv+\operatorname{Re}\sum_{m\neq m_0}\beta^1_m|\hat{P}_m|^2+\operatorname{Re}\sum_m\beta^2_m|\hat{T}_m|^2$$
$$+\operatorname{Re}\,w\iint_\Sigma|\mathcal{H}^N_t|^2\,d\tau=2\operatorname{Re}(\beta^1_{m_0}A\overline{P}^*_{m_0})-\operatorname{Re}\sum_m(\beta^1_mP^N_m\overline{P}^*_m+\beta^2_mT^N_m\overline{T}^*_m)$$
$$-k\ \operatorname{Im}\iiint_D\{(\vec{\varepsilon}\,\mathbf{E}^N)\mathbf{E}^{RN*}-(\vec{\mu}^*\mathbf{H}^{*N})\mathbf{H}^{RN}\}\,dv+\operatorname{Re}\left\{\iiint_D[(\operatorname{rot}\mathbf{H}^N)_t\mathbf{E}^{RN*}_t\right.$$
$$\left.+(\operatorname{rot}\mathbf{E}^{*N})_t\mathbf{H}^{RN}_t]\ dv\right\}-\operatorname{Re}\left\{\int dx_3\sum_{n=N+1}^{\infty}B_n\left[\iint_S E^N_{x_3}(\operatorname{rot}\mathbf{e}_n)_{x_3}\,d\tau\right]dl\right\}$$
$$+\operatorname{Re}\,w^{-1}\int dx_3\sum_{n=N+1}^{\infty}\oint(f^1_nA^*_n+f^{2*}_nB_n)\,dl. \tag{22}$$

It now remains only to show that the right-hand side of (22) tends to zero as $N\to\infty$. This indeed holds under the sole assumption that the exact solution of the problem belongs to the functional space L_2. The proof is carried out in complete analogy with the proof in works (^{1},^{2}).

Moscow State University
named after M. V. Lomonosov

Received
16 XI 1966

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Note: Figure translations are in progress. See original paper for figures.