ON A CLASS OF INTERPOLATION PROBLEMS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 512.52

MATHEMATICS

Yu. A. KAZMIN

ON A CLASS OF INTERPOLATION PROBLEMS

(Presented by Academician A. N. Kolmogorov on 8 IV 1966)

In what follows we shall use the following notation. The space of functions analytic in the disk $|z| < R$, $0 < R < \infty$, will be denoted by the symbol $A(|z| < R)$, and the space of functions analytic for $|z| \geq R$ and tending to zero at infinity will be denoted by $A(|z| \geq R)$. It is known that the spaces $A(|z| < R)$ and $A(|z| \geq R)$ are mutually conjugate. By the sign $[\rho, \sigma]$ below we denote everywhere the class of entire functions of growth not exceeding order ρ and type σ.

Let us first consider one interpolation problem of a particular nature, whose method of solution makes it possible to give an exhaustive answer for a certain class of interpolation problems.

Thus, let the following be given: 1) an integer p, $p \geq 1$; 2) p points $\eta\delta^k$, $\eta \neq 0$, $\delta = e^{2\pi i/p}$, $k = 0, 1, \dots, p-1$; 3) a point $\omega = e^{i\alpha}$ lying on the unit circle $|z| = 1$, where, for definiteness, $\alpha \in [0; 2\pi]$; and, finally, 4) p sequences of complex numbers

$$\{a_{ks}\}, \quad s = 0, 1, 2, \dots; \quad k = 0, 1, \dots, p-1.$$

The following questions are posed.

I. Does there exist an entire function $F(z) \in [1; \sigma]$ for which the relations

$$F^{(ps)}(\eta\delta^k\omega^s) = a_{ks}, \quad s = 0, 1, 2, \dots; \quad k = 0, 1, \dots, p-1. \tag{1}$$

hold?

II. If there exists a function $F(z) \in [1; \sigma]$ satisfying conditions (1), then how can $F(z)$ be reconstructed from the given values of its derivatives (1)?

III. What is the set of all functions of the space $A(|z| < R)$, $R > |\eta| > 0$, for which the equalities

$$F^{(ps)}(\eta\delta^k\omega^s) = 0, \quad s = 0, 1, 2, \dots; \quad k = 0, 1, \dots, p-1. \tag{2}$$

hold?

Questions of an analogous type are studied for the space $A(|z| \geq R)$.

IV. What is the set of all functions $F(z) \in A(|z| \geq R)$, $0 < R < |\eta|$, satisfying conditions (2)?

V. What properties must the numbers $\{a_{ks}\}$ have in order that there exist a function $F(z) \in A(|z| \geq R)$ corresponding to relations (1)?

VI. Suppose there exists a function $F(z) \in A(|z| \geq R)$ for which equalities (1) hold. It is asked how to reconstruct it from the known values of the derivatives (1).

Lemma 1. I. If, for the function

$$F(z) = \sum_{k=0}^{\infty} \frac{x_k}{k!} z^k \in A(|z| < R), \quad R > |\eta| > 0,$$

relations (1) are satisfied, then each of the auxiliary functions generated by it,

$$F_j(z) = \sum_{s=1}^{\infty} \frac{x_{ps-j}}{(ps-j)!} \omega^{(ps-j)(ps-j-1)/2p} z^{ps-j} \in A(|z| < R), \quad j = 1, 2, \dots, p,$$

in some neighborhood of the origin (in any case, for $|z| < R - |\eta|$) satisfies the corresponding linear differential equation of infinite order with constant coefficients

$$L_j[F_j] \equiv \sum_{s=1}^{\infty} \frac{\eta^{ps-j}}{(ps-j)!} \omega^{-(ps-j)(ps-j-1)/2p} F_j^{(ps-j)}(z) = \Phi_j(z), \qquad j = 1, 2, \dots, p, \tag{3}$$

where in (3) the right-hand side, for a given j, is the function

$$\Phi_j(z) = \sum_{s=0}^{\infty} \frac{B_{js}}{(ps)!} \omega^{s(ps-1)/2} z^{ps} \in A(|z| < r), \qquad r = R - |\eta| > 0, \tag{4}$$

whose coefficients B_{js} are defined by the equalities

$$B_{js} = \frac{1}{p} \sum_{q=0}^{p-1} \delta^{qj} a_{qs}, \qquad s = 0, 1, 2, \dots \tag{5}$$

II. If, for each j, $j = 1, 2, \dots, p$, there exists a solution $F_j(z) \in A(|z| < R)$, $R > |\eta| > 0$, of the j-th equation (3), whose right-hand side is determined by the series (4), with coefficients B_{js} computed by formulas (5), and the series (4) converges uniformly for $|z| < R - |\eta|$, while the j-th function $F_j(z)$, $j = 1, 2, \dots, p$, has a lacunary Taylor series in powers of z, containing only the powers z^{ps-j}, then for the function

$$F(z) = \Omega(z) * \left[\sum_{j=1}^{p} F_j(z)\right] \in A(|z| < R),$$

where

$$\Omega(z) = \sum_{s=0}^{\infty} \omega^{-s(s-1)/2p} z^s \in A(|z| < 1),$$

and the asterisk denotes the Hadamard product of the functions Ω and $\sum_{j=1}^{p} F_j$, the relations (1) are satisfied.

Recall that the Hadamard product of the functions $a(z) = \sum_{k=0}^{\infty} a_{kz}^k \in A(|z| < R_1)$ and $b(z) = \sum_{k=0}^{\infty} b_{kz}^k \in A(|z| < R_2)$ is the function

$$c(z) = a * b = \sum_{k=0}^{\infty} a_{kb_{kz}}^k \in A(|z| < R_1 R_2).$$

Renumber all zeros β_{nj} of the characteristic function

$$\varphi_j(\eta t) = \sum_{s=1}^{\infty} \frac{\eta^{ps-j}}{(ps-j)!} \omega^{-(ps-j)(ps-j-1)/2p} t^{ps-j} \in [1; \eta]$$

of the j-th equation (3) ($j = 1, 2, \dots, p$) in the order of nonincreasing moduli:

$$0 = |\beta_{0j}| < |\beta_{1j}| \le \dots \le |\beta_{nj}| \le |\beta_{n+1,j}| \le \dots;$$

moreover, for the function $\varphi_p(\eta t)$, which does not vanish at the origin, the list of roots in the displayed string begins with $|\beta_{1p}| > 0$. From the lacunary character of the expansion of the function $\varphi_j(\eta t)$ in powers of t it follows that if β_{nj}, $n = 1, 2, ...$, is its zero of multiplicity m_{nj}, then each of the p points

$$\{\beta_{nj}\delta^l\}, \qquad l = 0, 1, \dots, p-1 \quad (\delta = e^{2\pi i/p}) \tag{6}$$

is also a zero of the function $\varphi_j(\eta t)$ of the same multiplicity m_{nj}. For each group of roots of the form (6) we choose arbitrarily one representative $\hat{\beta}_{nj}$ and fix it.

Theorem A. I. In order that there exist an entire function $F(z) \in [1;\sigma]$ satisfying conditions (1), it is necessary and sufficient that the inequalities

$$\overline{\lim_{s\to\infty}} \sqrt[ps]{|a_{ks}|} \le \sigma, \qquad k = 0, 1, \dots, p-1.$$

II. The set of all entire functions of class $[1;\sigma]$ (σ fixed) for which the equalities (1) hold is the family of the form

$$F(z) = \sum_{j=1}^{p} \left[\frac{1}{2\pi i} \int_{|t|=\sigma+\varepsilon} \frac{\gamma_j(t)}{\varphi_j(\eta t)} \varphi_j(zt)\, dt + \sum_{0<|\tilde\beta_{nj}|\le\sigma} \sum_{l=0}^{m_{nj}-1} C_{njl} \frac{\partial^l}{\partial \tilde\beta_{nj}^l} \varphi_j(\tilde\beta_{nj} z) \right], \tag{7}$$

where the functions

$$\gamma_j(t) = \sum_{s=0}^{\infty} \frac{B_{js}\omega^{s(ps-1)/2}}{t^{ps+1}}, \quad j = 1, 2, \dots, p,$$

are regular for $|t| > \sigma^*$; the functions $\varphi_j(\eta t)$ and the numbers $\tilde\beta_{nj}, m_{nj}$ were defined above; the contour of integration $|t| = \sigma + \varepsilon$, $\varepsilon > 0$, is chosen so that in the annulus $\sigma < |t| \le \sigma + \varepsilon$ the functions $\varphi_j(\eta t)$ have no zeros; the sum in square brackets in representation (7) is taken over the zeros $\tilde\beta_{nj}$ of all groups of the form (6) chosen by us that fall in the circle $|t| \le \sigma$, and C_{njl} are arbitrary constants.

If, however, in the circle $|t| \le \sigma$ there are no zeros of the functions $\varphi_j(\eta t)$, $j = 1, 2, \dots, p$, then the second term on the right-hand side of equality (7) is identically equal to zero, and interpolation problem II has in the class $[1;\sigma]$ a unique solution.

III. Every function $F(z) \in A(|z| < R)$, $0 < |\eta| < R$, satisfying (2), is uniquely representable in the form of the sum of p series

$$\sum_{j=1}^{p} \sum_{n=1}^{\infty} \left(\sum_{l=0}^{m_{nj}-1} C_{njl} \frac{\partial^l}{\partial \tilde\beta_{nj}^l} \varphi_j(\tilde\beta_{nj} z) \right). \tag{8}$$

Moreover, a certain subsequence of partial sums of the sum of the series (8) converges uniformly in the circle $|z| < R - |\eta|$ to the function $F(z)$ (1^{-4}).

In what follows, for brevity, we shall say that the circle of questions I-III constitutes interpolation problem A, and questions IV-VI constitute problem B. Theorem A gives a complete solution of problem A. Note that for $p = 2$ and $\omega = 1$, item II of Theorem A solves a problem to which the well-known Lidstone problem on the reconstruction of an entire function $F(z) \in [1;\sigma]$ from the prescribed values of its even-order derivatives at two distinct points is reduced by a simple substitution: $F^{(2n)}(\alpha) = a_n,\ F^{(2n)}(\beta) = b_n,\ \alpha \neq \beta,\ n = 0, 1, 2, \ldots$ (see, for example, (5^{-8})).

By means of Lemma 1, problem A is reduced to a whole class of interpolation problems defined by prescribing relations of the form

$$F^{(ps-j)}(0) = a_s^{(j)}, \quad j = j_1, j_2, \ldots, j_n; \quad j_m \neq j_l,\ m \neq l; \quad 0 \leq j_m \leq p-1;$$

$$F^{ps}(\eta\delta_k\omega^s) = a_{ks}, \quad k = k_1, k_2, \ldots, k_{p-n}; \quad k_m \neq k_l,\ m \neq l; \quad 0 \leq k_m \leq p-1$$

$$(s = 0, 1, 2, \ldots).$$

Problem A is solved in an analogous way when $|\omega| < 1$ and $|\omega| > 1$.

Theorem B. I. Whatever may be: 1) an integer $p,\ p \geq 1$; 2) a complex $\eta,\ 0 < R < |\eta| < +\infty$; 3) a complex $\omega,\ |\omega| = 1$, there does not exist any function $F(z) \in A(|z| \geq R)$, different from the identically zero function, satisfying conditions (2).

II. In order that there exist a function $F(z) \in A(|z| \geq R)$ satisfying equalities (1), it is necessary and sufficient that the aggregate of complex numbers $\{a_{ks}\}$ generate functions

$$f_j^+(z) = \sum_{s=0}^{\infty} (-1)^{ps} \frac{\eta^{ps}\omega^{s(ps+j+2)}}{(ps)!} \left[\frac{1}{p}\sum_{k=0}^{p-1} a_{ks}\delta^{-k(j-1)}\right] z^{ps} \in A\left(|z| < 1 - \frac{R}{|\eta|} + \varepsilon\right),$$

$$\varepsilon > 0, \quad j = 1, 2, \ldots, p, \tag{9}$$

represented respectively by the series

$$f_j^+(z) = \sum_{s=1}^{\infty} C_{ps-j} \left[\frac{d^{ps-j}}{dt^{ps-j}}\left(\frac{t^{p-j}}{1-t^p}\right)\right]_{t=z/\omega^s}, \tag{10}$$

* The function $\gamma_j(t)$ is usually called the Borel-associated function of $\Phi_j(t) \in [1;\sigma]$ (4).

$$\overline{\lim_{s\to\infty}} \sqrt[ps-j]{(ps-j)!\,|C_{ps-j}|} < \frac{R}{|\eta|}, \qquad j = 1, 2, \ldots, p, \tag{10}$$

uniformly convergent for $|z| < 1 - R/|\eta| + \varepsilon$ $(\varepsilon > 0)$.

The representation (10), if it exists, is unique.

III. The Laurent coefficients x_n of the function

$$F(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^{n+1}} \in A(|z| \geq R),$$

satisfying the relations (1), are found from (10) by the formulas

$$x_{ps-j} = (ps-j)!\,\eta^{ps-j+1} C_{ps-j}, \qquad s = 1, 2, \dots; \quad j = 1, 2, \dots, p.$$

For each of the systems of functions

$$\left\{ \frac{d^{ps-j}}{dt^{ps-j}} \left(\frac{t^{p-j}}{1-t^p} \right) \Big|_{t=z/\omega^s} \right\}_{s=1}^{\infty},$$

considered in the space $A(|z| \geq 1 + R/|\eta|)$, a system of polynomials $\{p_{ps-j}(z)\}_{s=1}^{\infty}$ is constructed, containing only powers z^{ps-1} and forming, with the functions

$$\frac{d^{ps-j}}{dt^{ps-j}} \left(\frac{t^{p-j}}{1-t^p} \right) \Big|_{t=z/\omega^s},$$

a biorthogonal sequence. Therefore, if the functions $f_j^-(z)$, $j = 1, 2, \dots, p$, are known, to which, for $|z| > 1 + R/|\eta| - \varepsilon$, the series (10) respectively converge uniformly, then the coefficients C_{ps-j} (and then also x_{ps-j}) can be computed by the formulas

$$C_{ps-j} = \frac{1}{2\pi i} \int_{|t|=r>1+R/|\eta|} f_j^-(t)\, p_{ps-j}(t)\, dt, \qquad s = 1, 2, \dots; \quad j = 1, 2, \dots, p.$$

In connection with this, the following is of interest.

Theorem C. Let there exist a function

$$F(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^{n+1}} \in A(|z| \geq R),$$

satisfying the equalities (1), and let its elements $\{a_{ks}\}$ generate the functions (9)

$$f_j^+(z) \in A(|z| < 1 - R/|\eta| - \varepsilon), \qquad j = 1, 2, \dots, p,$$

analytically continuable from the disk $|z| \leq 1 - R/|\eta|$, respectively, along continuous curves Γ_j, $j = 1, 2, \dots, p$, without self-intersections, each of which connects the circles $|z| = 1 - R/|\eta|$ and $|z| = 1 + R/|\eta|$ and is located in the angle α_j (each in its own) with vertex at the origin and aperture not greater than $2(\pi - \arcsin R/|\eta|)$.

Then each of the functions (9)

$$f_j^+(z) \in A(|z| < 1 - R/|\eta| + \varepsilon), \qquad j = 1, \dots, p,$$

can be analytically continued to the whole domain $|z| \geq 1 + R/|\eta|$, and the result of continuing the function $f_j^+(z)$ along Γ_j into the domain $|z| \geq 1 + R/|\eta|$ coincides respectively with

$$f_j^-(z) \in A(|z| \geq 1 + R/|\eta|), \qquad j = 1, 2, \dots, p.$$

To the interpolation problem B considered in the space $A(|z| \geq R)$ there reduces any of the interpolation problems of the following type: construct a function

$$F(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^{n+1}} \in A(|z| \geq R),$$

if the following are known: 1) its Laurent coefficients x_{ps-j}, $s = 1, 2, \dots;$ $j = j_1, j_2, \dots, j_m$, $j_n \neq j_l$, $n \neq l$; $1 \leq j_n \leq p$, $1 \leq n \leq p$; 2) the values of the derivatives

$$F^{(ps)}(\eta\delta^k\omega^s) = a_{ks}, \qquad s = 0, 1, \dots; \quad k = k_1, k_2, \dots, k_{p-m}, \ k_n \neq k_l, \ n \neq l, \ 0 \leq k_n \leq p-1$$

$(|\eta| > R)$. With the corresponding modifications, the method proposed by us makes it possible to solve problems B of the indicated type also in the case when $|\omega| > 1$.

Moscow State University
named after M. V. Lomonosov

Received
8 IV 1966

References

1. A. O. Gel' fond, *Calculus of Finite Differences*, Moscow, 1951.

2. A. O. Gel' fond, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, **38**, 42 (1951).

3. A. F. Leont'ev, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, **39** (1951).

4. R. D. Carmichael, Bull. Am. Math. Soc., **42**, 193 (1936).

5. J. M. Whittaker, Proc. Lond. Math. Soc., **36**, 451 (1933–1934).

6. R. P. Boas, Duke Math. J., **10**, 234 (1943).

7. R. C. Buck, Trans. Am. Math. Soc., **64**, 283 (1948).

8. R. F. De Mar, Trans. Am. Math. Soc., **114**, 23 (1965).

Note: Figure translations are in progress. See original paper for figures.