# THE RIEMANN–HILBERT BOUNDARY-VALUE PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS FOR QUASILINEAR ELLIPTIC SYSTEMS OF EQUATIONS

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## Abstract

## Full Text

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**MATHEMATICS**

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# THE RIEMANN–HILBERT BOUNDARY-VALUE PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS FOR QUASILINEAR ELLIPTIC SYSTEMS OF EQUATIONS

*(Presented by Academician M. A. Lavrent′ev on 26 IX 1966)*

Consider the general quasilinear elliptic system of first-order equations, which can be written in the form of one complex equation

$$\omega_{\bar z} - \mu_1(z,\omega)\omega_z - \mu_2(z,\omega)\overline{\omega}_{\bar z} + F(z,\omega) = 0, \qquad (1)$$

where $\omega = u + iv$, $z = x + iy$, $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar z = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$; $\mu_1(z,\omega)$, $\mu_2(z,\omega)$, and $F(z,\omega)$ satisfy a Lipschitz condition with respect to $\omega$ for each fixed $z \in \overline{K}$ (the domain $K$ may, without loss of generality, be assumed to be the disk $|z| \le 1$) and are measurable with respect to $z$ for fixed $\omega$.

The condition of uniform ellipticity of the system (1) can be written in the form

$$|\mu_1(z,\omega)| + |\mu_2(z,\omega)| \le \mu_0 < 1 \qquad (2)$$

for $z \in \overline{K}$ and arbitrary $\omega$.

The Riemann–Hilbert problem for the system (1) consists in finding a solution of this system satisfying the boundary condition

$$\operatorname{Re}\{(a+ib)\omega(t)\}\big|_\Gamma = \gamma(t), \qquad t = e^{i\gamma}, \quad \gamma \in [0, 2\pi]. \qquad (3)$$

In the case when the coefficients of the boundary condition satisfy the Hölder condition ($a(t) \in H_\alpha$, $b(t) \in H_\alpha$), the problem (1), (3) was studied in the work of V. S. Vinogradov ([1]). In the works of V. N. Monakhov ([2,3]), in connection with problems of gas dynamics, one particular problem (1), (3) (with $F = 0$) was studied in the case of discontinuous coefficients $a(t)$ and $b(t)$, the so-called mixed boundary-value problem, when

$$a + ib = \begin{cases} 1, & \gamma \in [0, \pi], \\ i, & \gamma \in [\pi, 2\pi]. \end{cases}$$

In the present paper the general Riemann—Hilbert boundary-value problem (1), (3) with discontinuous coefficients is studied in the case $F = 0$, to which problems of gas dynamics with free boundaries lead ([2,3]). Thus, let $a(t)+ib(t) \in H_\alpha$ for $t = [t_k, t_{k+1}]$ $(k = 1, \ldots, m-1)$ and

$$a(t_k - 0) + ib(t_k - 0) \neq a(t_k + 0) + ib(t_k + 0).$$

By a change of the unknown functions one can reduce the problem to the case

$$\{a(t) + ib(t)\} = a_k + ib_k = \text{const}, \qquad t \in [t_k, t_{k+1}].$$

Then the canonical function solving the homogeneous problem (3) in the class of analytic functions bounded at all points can be represented in the following form:

$$X(z) = \prod_{k=1}^{m} (z - t_k)^{\gamma_k} z^n = R(z) z^n, \tag{4}$$

where $0 \leq \operatorname{Re} \gamma_k < 1$, and $n$ is an integer, positive or negative.

Consider problem (1), (3) in the following formulations.

**Problem I** $(n \geq 0)$. Find a solution $\omega \in W'_{p,\, p>2}$ of equation (1) satisfying the boundary condition (3) and the relations

$$\int_\Gamma \frac{\omega - \Phi}{R} \, z^{-k} \, ds = 0 \qquad (k = 0, \ldots, 2n), \tag{5}$$

where $\Phi = \Phi(z)$ is an analytic function satisfying condition (3).

**Problem II** $(n < 0)$. Find a solution $\omega \in W'_{p,\, p>2}$ of equation (1) satisfying the boundary condition (3).

Both problems reduce to singular integral equations, for which unique solvability is proved.

We shall seek the solution of boundary-value problems I, II in the form

$$\omega^i(z) = T_i f_i + \Phi_i(z) \qquad (i = 1, 2), \tag{6}$$

where $\Phi_1(z)$ is analytic, while $\Phi_2(z)$ is a function conjugate to an analytic one, solving the inhomogeneous problem (3) in the class of bounded functions. The operators $T_i$ have the form

$$T_1 f = -\frac{R(z)}{\pi} \iint_K \left\{ \frac{f(t)}{R(t)(t-z)} + \frac{\overline{f}(t) z^{2n+1}}{\overline{R}(t)(\bar{t}z-1)} \right\} dK, \tag{7}$$

$$T_2 f = -\frac{R(z)}{\pi} \iint_K \left\{ \frac{f(t)}{R(t)(t-z)} + \frac{\overline{f}(t)\, \bar{t}^{\,2|n|-1}}{\overline{R}(t)(\bar{t}z-1)} \right\} dK. \tag{8}$$

Using the results of papers ([2,3]), we prove that the operators $T_i$ are completely continuous in $L_p$ $(p > 2)$ and

$$\frac{\partial}{\partial \bar{z}} T_i f = f,$$

while the operators

$$S_i f \equiv \frac{\partial}{\partial z} T_i f = S_i^0 f + T_i^0 f$$

are linear and bounded in $L_p$ $(p > 2)$; moreover

$$T_1^0 f = \frac{(2n+1)R(z)z^n}{\Lambda} \iint_K \frac{\overline{f}(t)\, dK}{\overline{R}(t)(1-\bar{t}z)}$$

is completely continuous in $L_p$, while $T_2^0 f = 0$ and

$$\|S_i^0\|_{L_2} = 1.$$

The function $\omega = T_1 f$ satisfies the homogeneous condition (3) for any $f \in L_p$, while the function $\omega = T_2 f$ satisfies this homogeneous condition for $n < 0$ only when, for $f \in L_p,\ p > 2$, the relations

$$a_j(f) = -\frac{1}{\pi} \iint_K \left\{ \frac{f(t)}{R(t)}\, t^{j-1} + \frac{\overline{f}(t)}{\overline{R}(t)}\, \bar{t}^{\,2n-j-1} \right\} dK = 0 \qquad (j = 1, \dots, n) \tag{9}$$

are fulfilled.

Substituting $\omega_i$ into equation (1), we obtain the singular equations

$$
\begin{aligned}
&f_i - \mu_1(z, T_i f_i + \Phi_i) S_i f_i - \mu_2(z, T_i f_i + \Phi_i)\overline{S_i f_i} \\
&\quad = \mu_1(z, T_i f_i + \Phi_i)\, d\Phi_i/dz + \mu_2(z, T_i f_i + \Phi_i)\, d\overline{\Phi_i}/d\bar{z} - d\Phi_i/d\bar{z} \\
&\qquad (i = 1, 2, \quad d\Phi_1/d\bar{z} = d\Phi_2/dz = 0).
\end{aligned} \tag{10}
$$

In view of condition (2) and the continuity of $\|S_i\|_{L_p}$ with respect to $p$, there exists a

$$p = 2 + \delta(p), \qquad \delta > 0,$$

such that

$$\|\mu_i(z, \omega^i) S_i f_i + \mu_2(z, \omega^i)\overline{S_i f_i}\|_{L_p} \leq k\|f_i\|_{L_p} \qquad (k < 1). \tag{11}$$

Thus, to equation (10) for $i = 2$ we apply the principle of contractions, and consequently equation (10) has a unique solution for $n < 0$, which is the solution of the original problem II under fulfillment of the additional solvability conditions (5).

Consider equation (10) for $i = 1,\ n \geq 0$. In view of inequality (11), this equation is reduced to the form

$$f_1 + Bf_1 = 0, \tag{12}$$

where

$$Bf_1 = [J - \mu_1 S_1^0 - \mu_2 \overline{S_1^0}]^{-1} \times [\mu_0 d\Phi/dz + \mu_2 d\overline{\Phi}/d\bar{z} + \mu_1 T_1^0 f + \mu_2 \overline{T_1^0 f}).$$

Analogously to paper (2), the complete continuity of the operator $B$ is established. Inequality (11) and the uniform ellipticity of equation (1) make it possible to obtain a strong a priori estimate of the solution

$$\|f\|_{L_p} \leq M\left(p, \mu_0, \|d\Phi/dz\|_{L_p}\right).$$

From the complete continuity of the operator $B$ and the above a priori estimate there follows the applicability to equation (12) of Schauder' s fixed-point principle. Thus, the existence of at least one solution of equation (10), and hence of problem II, has been proved. Let us show that the solution of problem II is unique. Analogously to paper (2), it is shown that the difference of two solutions $\omega_1 - \omega_2 = \omega[z(\zeta)]$, as a function of $\zeta$ ($\zeta = \varphi(z)$ is some fixed homeomorphism of the disk onto the disk), is a generalized analytic function for which the well-known representation (4) holds:

$$\omega[z(\zeta)] = \psi(\zeta)\exp\left\{-\frac{1}{\pi}\iint_K \left\{\frac{\rho[\omega_1(t),\omega_2(t)]}{t-\zeta} + \frac{\overline{\rho[\omega_1(t),\omega_2(t)]}\,\zeta}{\overline{t_\zeta}-1}\right\} dK\right\},$$

where $\rho \in L_p,\ p > 2$, is a certain fixed function. The function $\varphi(\zeta)$, analytic in the disk $|\zeta| < 1$, must by construction satisfy the homogeneous boundary condition (3) and the homogeneous supplementary conditions of problem I (all the conditions, of course, transformed to the plane of the homeomorphism $\xi = \zeta(z)$), which is possible only when $\psi \equiv 0$. Thus it is proved.

**Theorem.** *Problem I is uniquely solvable, while problem II has a unique solution only when the additional solvability conditions (5) are satisfied.*

This theorem also holds for equation (1) when $F \not\equiv 0$, but its proof requires obtaining more complicated a priori estimates of the solution.

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*Note: Figure translations are in progress. See original paper for figures.*