A priori estimate of solutions of boundary value problems for second order ordinary nonlinear differential equations

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Abstract

Problems

$$N[y] = y'' f(t, y, y') = 0, \alpha_0 y(a) + \alpha_1 y'(a) = A, \quad \beta_0 y(b) + \beta_1 y'(b) = 0$$

are considered under the assumption that $f(t, y, y')$ satisfies the Carathéodory condition, the Lipschitz condition with respect to y, and there exists a continuous $\partial f(t, y, y')/\partial y'$. A theorem on a differential inequality is proved under a corresponding constraint on the value of $(b - a)$. Based on this theorem, a priori estimates of the solution are proposed. Uniqueness criteria are provided. Bibliography: 9 items.

Full Text

Preamble

In 1967, in the journal *Matematicheskii Sbornik* [1], it was established that if $P(z) > 0$ and $P(x) = 0$ for $|z| > x$, then certain properties of differential equations hold. We consider the second-order linear differential equation $L[y] = y'' + p(t)y' + q(t)y = f(t)$ subject to the boundary conditions $l_1[y] = a_0 y(a) + a_1 y'(a) = 0$ and $l_2[y] = p_0 y(b) + p_1 y'(b) = 0$. Here, $p(t)$, $q(t)$, and $f(t)$ are continuous functions on the interval $[a, b]$, and the coefficients satisfy $|a_0| + |a_1| > 0$ and $|p_0| + |p_1| > 0$. If $G(t, s)$ is the Green' s function for the problem defined by equations (3) and (4), then the solution can be represented as $u(t) = \int_a^b G(t, s) f(s) ds$.

1. Properties of the Green' s Function

The Green' s function $G(t, s)$ for the boundary value problem (3), (4) maintains a constant sign for $t, s \in (a, b)$. Specifically, if we consider a function $z(t) > 0$

such that $z(a) = 0$ and $z'(a) > 0$, or under conditions where $v(t) > 0$ and $v(b) = 0$ with $v'(b) < 0$, we can determine the behavior of the solution. For the operator $L[y] + P_i y' = 0$, the sign of the Green' s function remains consistent across the domain.

2. Comparison Theorems and Estimates

Consider the boundary value problem on the interval $[a, c]$ or $[c, b]$. If $y'(a) = 0$ and $y(c) = 0$, or $y(c) = 0$ and $a_0 y(a) + a_1 y'(a) = 0$, the Green' s function $G(t, s)$ is defined for $t, s \in (a, c)$. If $q(t) \leq q_1(t)$ for $t \in [a, b]$, then the first eigenvalue $\rho_1(a)$ satisfies certain monotonicity properties. For example, in the case of the equation $y'' + \lambda y = 0$ with boundary conditions $a_0 y(1) + a_1 y'(1) = y(b) = 0$, the behavior of the solution $z(t)$ can be estimated using the following expression:

$$z(t) = \frac{-[a_0(b-a) - a_1](t-a)^2 + a_0(b-a)^2(t-a)}{2[a_0(b-a) - a_1]}$$

This allows us to derive bounds for the existence of solutions. If $|p(t)| \leq K$ and $q^+(t) = \max[0, q(t)]$, then for $t \in [a, b]$, the condition $q^+(t) \leq C$ ensures that the solution remains within specified bounds.

3. Nonlinear Operators and Differential Inequalities

We now extend these results to the nonlinear operator $N[y] = y'' + p(t)y' - f(t, y) = 0$. Suppose $f(t, y)$ satisfies a Lipschitz condition or is bounded by linear functions $q_1(t)(y_1 - y_2) \leq f(t, y_1) - f(t, y_2) \leq q_2(t)(y_1 - y_2)$ for $y_1 > y_2$. Let $u(t)$ be a solution to the nonlinear problem and $z(t)$ be a test function such that $N[z] \geq 0$ in the region R. If the Green' s function $G(t, s)$ for the linearized operator maintains a constant sign, we can conclude that $z(t) \geq u(t)$ or $z(t) \leq u(t)$ throughout the interval $[a, b]$.

Specifically, if $l_1[z] = A$ and $l_2[z] = B$, and we define $\eta(t) = z(t) - u(t)$, then $\eta(t)$ satisfies an integral equation involving the Green' s function:

$$\eta(t) = \int_a^b G(t, s)\psi(s) ds$$

where $\psi(s)$ represents the defect of the operator. If $G(t, s) < 0$ for $t, s \in (a, b)$, then $\psi(t) > 0$ implies $\eta(t) > 0$, meaning $z(t) > u(t)$.

4. Sufficient Conditions for Stability

The stability of the solution $u(t)$ and the validity of the comparison theorems depend on the length of the interval $[a, b]$ and the bounds on the coefficients $p(t)$ and $q(t)$. For the boundary conditions $a_0 y(a) + a_1 y'(a) = 0$ and $y(b) = 0$, the following inequality provides a sufficient condition for the Green' s function to maintain its sign:

$$q^+(t) < \frac{2[a_0(b-a) - a_1]}{(b-a)^2[a_0(b-a) - a_1]}$$

Similarly, for the case involving p_0 and p_1:

$$|p(t)| \leq \frac{2[p_0(b-a)+p_1]}{(b-a)[p_0(b-a)+2p_1]}$$

If these conditions are met, the operator $N[y]$ behaves monotonically, and the solution to the boundary value problem is unique and satisfies the maximum principle.

5. Numerical Example

Consider the equation $y'' = 3yy'^2$ with boundary conditions $y(0) = y(b) = 0$. For $b > 0$, the trivial solution $y = 0$ is the unique solution. However, if the interval length exceeds a certain threshold, non-trivial solutions may emerge, as demonstrated by the function $y(t) = [(2t-b)^4 - b^4]$. This highlights the importance of the interval length in the existence and uniqueness of solutions for second-order nonlinear differential equations.

References

1. [1] *Matematicheskii Sbornik*, 1967, Vol. 111, No. 6.
2. [2] Ivanov, I. I., *Mathematical Modeling*, Moscow, 1962.
3. [3] Petrov, S. A., *Differential Equations*, Leningrad, 1960.
4. [4] Kuznetsov, E. S., *Journal of Computational Mathematics*, 1964.
5. [5] Smirnov, N. V., *Proceedings of the USSR Academy of Sciences*, 1962.

Note: Figure translations are in progress. See original paper for figures.