The asymptotic equivalence of the solutions of certain linear systems of differential equations

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Abstract

The question of the equivalence of systems of two differential equations [?]

$$\frac{dx}{dt} = X[R(t) + Q(t)], \tag{1}$$

$$\frac{dx}{dt} = YR(t) \tag{2}$$

in the sense of $x = YA(t)$, where $A(t) \to A = \text{const}$ as $t \to \infty$, is considered. Here $R + Q = P_0 + P_1 t^{-1} + P_2 t^{-2} + \cdots$, where P_k are constant second-order matrices, and $R(t)$ is a segment of this series. Bibliography: 7 items.

Full Text

Preamble

In this section, we investigate the asymptotic behavior of the solutions to the system of differential equations as $t \to \infty$. We consider the relationship between the matrices $R(t)$, $Q(t)$, and X, specifically focusing on the case where $X' = X[R(t) + Q(t)]$ and $Y' = YR(t)$. We assume that $Y^{-1}X = A(t)$, where $A(t) \to A = \text{const}$ as $t \to \infty$.

§ 1. Case of Distinct Real Eigenvalues

Let the matrix P_0 have distinct real eigenvalues α_1 and α_2, with $\alpha_1 > \alpha_2$. Suppose the matrix $R(t)$ can be expanded in a series of the form:

$$R(t) = P_0 + P_1 t^{-1} + P_2 t^{-2} + \cdots = \sum_{k=0}^{\infty} P_k t^{-k} \qquad (1.1)$$

where P_k are constant matrices. Following the methods established in [?], the fundamental matrix Y can be represented as:

$$Y = e^{\text{diag}[\alpha_1, \alpha_2] t} Z(t) \qquad (1.2)$$

where $Z(t) = I + O(t^{-1})$ as $t \to \infty$. If we define $A(t) = Y^{-1}X$, and assume $A(t) \to A = \text{const}$, then the matrix A must be diagonal, $A = \text{diag}[b_{11}, b_{22}]$. If $b_{11}b_{22} \neq 0$, then the matrix X exhibits the same asymptotic growth as Y.

§ 2. Case of Purely Imaginary Eigenvalues

Consider the case where P_0 has purely imaginary eigenvalues $P_0 = \text{diag}[i\alpha, -i\alpha]$. If the first-order perturbation term P_1 vanishes ($P_1 = 0$), the asymptotic behavior is determined by higher-order terms. Under these conditions, the transformation $Y = e^{\text{diag}[i\alpha, -i\alpha]t} Z(t)$ yields a matrix $A(t)$ that approaches a constant diagonal matrix $B = \text{diag}[b_{11}, b_{22}]$. If $b_{11}b_{22} \neq 0$, the system remains stable, and the solutions are bounded.

§ 3. Case of Complex Conjugate Eigenvalues

When P_0 has complex conjugate eigenvalues of the form $\alpha \pm i\alpha$, we utilize the representation:

$$Y = e^{\alpha t} e^{\text{diag}[i\alpha, -i\alpha]t} Z(t) \qquad (3.1)$$

The analysis follows a similar logic to the previous sections. The matrix $A(t)$ converges to a constant matrix A as $t \to \infty$. The specific form of A depends on the coefficients P_k of the expansion of $R(t)$. If $P_1 = 0$, the convergence rate is determined by $O(t^{-2})$.

§ 4. Case of Multiple Eigenvalues

In the case where P_0 has a multiple eigenvalue α with a non-trivial Jordan block:

$$P_0 = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \qquad (4.1)$$

the asymptotic expansion of the solution X involves logarithmic terms. Specifically, the matrix Y can be expressed as:

$$Y = e^{\alpha t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} Z(t) \qquad (4.4)$$

where $Z(t) = I + O(t^{-1})$. If $A(t) = Y^{-1}X \to A$, then A must commute with the Jordan form of P_0.

For the general case where $R(t) = P_0 + \sum_{k=1}^{\infty} P_k t^{-k}$, we define the matrices L_k based on the coefficients P_k. If the determinant $D(S) \neq 0$ and the structural conditions on the elements s_{21}, s_{22} are met, the solution X can be represented as:

$$X = t^C Z(t) S^{-1} \text{diag}[t, 1] e^{\alpha t} \qquad (4.18)$$

where C is a constant matrix related to the eigenvalues of the perturbation.

If the leading perturbation term P_1 satisfies certain nullity conditions, the logarithmic growth is suppressed, and $A(t)$ converges to a constant matrix B more rapidly. Specifically, if $b \neq 0$ and the resonance conditions are avoided, the solution maintains the form:

$$A(t) = \begin{pmatrix} t^{-1} & -\ln t \\ 0 & 1 \end{pmatrix} Z^{-1}(t) B X Z(t) S^{-1} \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix} \tag{4.37}$$

This indicates that the interaction between the Jordan structure of the unperturbed system and the power-law decay of the perturbation $R(t) - P_0$ leads to a variety of asymptotic regimes, including pure power-law growth, logarithmic corrections, or convergence to a steady state.

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Figures

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CLASSIFICATION OF TRAJECTORIES OF A DYNAMICAL SYSTEM WITH A CYLINDRICAL PHASE SPACE

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1. The study of oscillations of pendulum systems (simple pendulum, system of coupled pendulums, double pendulum, etc.), as well as the study of the dynamics of electromechanical systems, inertial systems of television synchronization, phase-locked loops [1], leads to the necessity of considering a system of differential equations of the form

$$\begin{aligned} d\varphi_i/dt &= \Phi_i(\varphi_1, \dots, \varphi_m, x_1, \dots, x_n) \quad (i = 1, \dots, m), \\ dx_j/dt &= X_j(\varphi_1, \dots, \varphi_m, x_1, \dots, x_n) \quad (j = 1, \dots, m), \end{aligned} \tag{1}$$

where the variables $\varphi_1, \dots, \varphi_m$ are angular (phase) coordinates and the functions Φ_i, X_j are periodic functions (with period 2π) of these coordinates, the variables $x_1, \dots, x_n$ are Euclidean coordinates.

Without loss of generality, we can assume that the period of all angular coordinates is the same and equal to 2π. This means that the physical state of the system under consideration, described by points of the form $(\varphi_1 + 2k_1\pi, \dots, \varphi_m + 2k_m\pi, x_1, \dots, x_n)$, where $k_1, \dots, k_m$ are integers, is identical. By identifying all points of the indicated type, we obtain a cylindrical phase space $R(\varphi_1, \dots, \varphi_m)$. This space can be imagined geometrically as the topological product of an m-dimensional torus and an n-dimensional Euclidean space of variables $x_1, \dots, x_n$.

The Euclidean space R of variables $\varphi_1, \dots, \varphi_m, x_1, \dots, x_n$ will be a covering space for the cylindrical space $R(\varphi_1, \dots, \varphi_m)$ (see, for example, [2] and [3]).

The cylindrical space $R(\varphi_1, \dots, \varphi_m)$ can be obtained from the space R, if the latter is cut along the surfaces $\varphi_i = -\pi$, $\varphi_i = \pi$ $(i = 1, \dots, m)$ and the obtained "strip" is glued along the cut surfaces. It is clear that such a rolling up can be carried out not for all coordinates $\varphi_1, \dots, \varphi_m$, but only for a certain part of these coordinates, for example, for the coordinates $\varphi_r, \dots, \varphi_s$; the cylindrical space obtained in this way will be denoted by the symbol $R(\varphi_r, \dots, \varphi_s)$. Obviously, the space $R(\varphi_r, \dots, \varphi_s)$ is also a covering for the space $R(\varphi_1, \dots, \varphi_m)$.

Assuming that some conditions ensuring the existence and extendability of solutions to the system of equations (1) are satisfied, we obtain a dynamical system on the phase space $R(\varphi_1, \dots, \varphi_m)$. This dynamical system induces in any of the covering spa-

Figure 1: Figure 1

is conducted using a continuous deformation into another. The number of different independent classes of closed paths is called the order of connectivity of the manifold. The order of connectivity of the space $R(\varphi_1, \ldots, \varphi_m)$ is equal to $m+1$, thus, the maximum number of homotopy independent limit cycles in the space $R(\varphi_1, \ldots, \varphi_m)$ is $m+1$.

Such a maximum system can be represented, for example, by a system consisting of a 0-cycle and φ_i-cycles, where $i = 1, \ldots, m$.

Thus, the proposed classification of limit cycles gives a finer partition into classes compared to the classification based on the concept of homotopy, since all cycles of a class different from class (0) and classes (φ_i), $i = 1, \ldots, m$, will be homotopy dependent on the above-mentioned limit cycles. Nevertheless, our classification, being amnvre simple,, allows for a more accurate charscterization of the location of a limit cycle in the considered cylindrical phase space. So, for example, on a two-dimensional torus $R(\varphi_1, \varphi_2)$ we have three homotopy independent classes of limit cycles: 0-cycles, φ_1-cycles, encompassing the torus along the meridian, φ_2-cycles, encompassing the torus along the parallel. Obviously, (φ_1, φ_2)-cycles will encompass the torus along the meridian and along the parallel, i.e. despite the fact that they are derivatives of the cycles indicated above, they can represent an independent interest in the study.

3. As property A, any other property of the trajectories of the dynamical system can serve. One of such properties can serve, for example, the prop--erty of positive stability of trajetories according to Poisson ([S], p. 363).

Recall that a point p is called positively stable according to Poisson, if for any neighborhood U of this point one can indicate a positive number T such that during its movement along the trajectory within the time interval $i \geq T$ the point p at least once falls again into the neighborhood U. It is known that if at least one point of a trajectory is positively stable acosson will be known that if it at mont one point of a trajectory is positively stable according to Poision, then all other points of this trajectory will possess the same property, thus. Thus, we obtain the concept of a positively stable according to Poisson trajectory (P-stable trajectory).

Following definition p. 2. we say that trajectories from the space $R(\varphi_1, \ldots, \varphi_m)$ are of P-stable class $(\varphi_j, \ldots, \varphi_s)$, if the property of P-stability is preserved under deployment along all coordinates φ_j, where j is different from $r, \ldots, s$, and disappears under further deployment along any of the coordinates $\varphi_r, \ldots, \varphi_5$.

If as the space $R(\varphi_1, \varphi_2)$ we take a two-dimensional torus, then the only P-stable trajectories on it, different from rest points and limit cycles, will be P-stable trajectories of class (φ_1, φ_2). In fact, the deployment of the torus along any of the coordinates φ_1, φ_2 is a cylinder, and along both coordinates $\varphi_{\prime\prime}, \varphi_2$ - a plane. Not on the plane, nor on the two-dimensional cylinder can there be P-stable trajectories, different from rest points and limit cycles ([S], p. 201).

Thus, on the torus there exist only P-stable trajectories of class (φ_1, φ_2), different from special points and limit cycles. The property of P-stability of these trajectories is lost upon deployment along any of the coordinates φ_1, φ_2.

If as the space $R(\varphi_1, \varphi_2)$ we take the topological product of a torus and the real line, i.e. we introduce thene we introduce into consideration a new coordinatod x, which is not angular, then the deployments of such a space

Figure 2: Figure 2

spacces will by three-dimensional cyindees, and the full unfolding – a three-dimensional Euclidean spacies. Obviously, in this case it is not difficult to construct examples of P-stohvive trajectories, different of singular cycles and limitament cycles, of as any of the fetuips possible classes.

Theorem 1. Let there exist in the space $\mathbf{R}$ a single-valued continuously differentiable scalar function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$, the derivative of which, is taken in virtue of the system (1), is sign-constant. Let the function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ be periodic (with period 2π) in the coordinates $\varphi_r, \ldots, \varphi_s$. Then all P-stable trajectories of class ($\varphi_r, \ldots, \varphi_s$) (as well as of class (0) and of class (0) and of class ($\varphi_p, \ldots, \varphi_q$), where $\varphi_p, \ldots, \varphi_q$ – coordinates from the set $\varphi_r, \ldots, \varphi_s$) lie on the surface $\dot{v} = 0$.

Thus, if you can find $\dot{v} = 0$ containing no entire trajectories, then P-stable trajectories and $\dot{v} = 0$ containing no entire trajectories, then P-stable trajectories of the classes, indicated in the theorem are absent.

There is a proof of the theorem, let us consider the space $\mathbf{R}(\varphi_r, \ldots, \varphi_s)$, which is obtained from the space $\mathbf{R}$ by adding along the coordinates $\varphi_r, \ldots, \varphi_s$. Also suppose, the function $v(\varphi_1, \ldots, \varphi_m, x_m, x_1, \ldots, x_n)$ is single-valued and continuous in the space $\mathbf{R}(\varphi_r, \ldots, \varphi_s)$, as well as the associated function at corresponding points upon adding will be identical.

Let us now assume for definiteness, the function $\dot{v}(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ is sign-constant in the space $\mathbf{R}$, i.e. satisfies everywhere the inequality $\dot{v} \leq 0$.

Let us further suppose, that there exists a P-stable trajectory of class ($\varphi_r, \ldots, \varphi_s$). By definition, this trajectory answers P-stable in space $\mathbf{R}(\varphi_r, \ldots, \varphi_s)$ and has the same property upon unfolding along any angular coordinate $\varphi_r, \ldots, \varphi_s$.

Let us assume that on our trajectory exists at least one point q, at which $\dot{v}$ is different from zero, i.e. non-zero. By definition of P-stability for any positive number T it is possible to indicate a number $t > T$ such, that after time t the point q arrives in an arbitrarily given neighborhood of its initial position. This means, that the function v with growth of time must take values, arbitrarily close to $v(q)$. But the latter leads to a contradiction, since as a non-increasing function v along the considered trajectory, moreover by the choice of the point q the value of this function with growth of time will do even less value, ascribed to function v at the point q.

5. Let us now try to obtain a more general result. Let us consider again the cylindrical space $\mathbf{R}(\varphi_1, \ldots, \varphi_m)$.

Consider a positive semi-trajectory of some point p and a sequence of positions $p(t_n)$ of this point upon movement along the trajectory of system (1), corresponding to the moments of time $0 < t_1 < t_2 < \ldots, t_n \to \infty$. By definition, any point q, limiting for the set $\{p(t_n)\}$, is called an ω-limit point of the point p. As is known ([5], p. 358), the set ω-limit points of a given point p is closed and invariant, i.e. it consists of entire trajectories.

We will now say, that the point q is an ω-limit point of class ($\varphi_r, \ldots, \varphi_s$) for the point p, if the point q is an ω-limit for p in the space $\mathbf{R}(\varphi_r, \ldots, \varphi_s)$ and has the property upon unfolding along any of the coordinates.

Figure 3: Figure 3

Theorem 2. *Let there exist in the space $\mathbb{R}$ a continuously differentiable single-valued scalar function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$, whose derivative, taken in virtue of system (1), is of constant sign. If the function v is periodic with respect to the coordinates $\varphi_r, \ldots, \varphi_s$ (with period 2π), then all ω-limit points of class $(\varphi_r, \ldots, \varphi_s)$ lie on the set $v = 0$.*

Indeed, as before, we are convinced that in the space $\mathbb{R}(\varphi_r, \ldots, \varphi_s)$ the function v will be a single-valued function. Since the derivative of the function v is of constant sign, then with increasing time v monotonically changes along the trajectory and has a finite or infinite limit v_0 as $t \to$ $t \to \infty$. But it is easy to see that in the case of an infinite limit, ω-limit points of the trajectory under consideration will be absent. Let us the therefore concentrate on the case when v_0 is a finite quantity. If point q is any ω-limit point of the given trajectory, then from the continuity continuity and monotonic character of the change of function v along the trajectory it change of function v along the trajectory it follows that $v(q) = v_0$. Thus, the entire ω-limit set of the trajectory lies on the same level surface $v = v_0$. Since the ω-limit set consists of entire trajectories, then along these trajectories we have $v = 0$, which proves the assertion of the theorem.

Since the points of a trajectory positively stable according to Poisson are ω-limit points for this trajectory, it is not difficult to obtain Theorem 1 as a consequence of Theorem 2. Theorem 2 resembles in its formulation Lemma 5.1 and Theorem 5.2 from work [6], as well as the more general LaSalle theorem [7].

A trajectory is called positively stable according to Lagrange (L-stable) if the closure of any positive semi-trajectory is compact. Analogously to the previous, one can give a definition of L-stability of class $(\varphi_r, \ldots, \varphi_s)$. Obviously, the set of ω-limit points of class $(\varphi_r, \ldots, \varphi_s)$ for an L-stable trajectory of class $(\varphi_r, \ldots, \varphi_s)$ is not empty – class e→ empty. From the proof of Theorem 2 it follows that under the conditions of this theorem, any L-stable point of class $(\varphi_r, \ldots, \varphi_s)$ approaches indefinitely as $t \to \infty$ in the space $\mathbb{R}(\varphi_r, \ldots, \varphi_s)$ to nhich to some invariant set invariant set lying on the set $v = 0$. If in this case the set $v = 0$ in the space $\mathbb{R}(\varphi_r \ldots \mathbb{R}(\varphi_r, \ldots, \varphi_s)$ consists of only one point O, then manty L-stable point of this space asymptotically tends as $t \to \infty$ to this point O.

6. Consider now in the space $\mathbb{R}$ the system

$$\begin{aligned} \frac{d\varphi_i}{dt} &= \Phi_i(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n) \quad (i = 1, \ldots, m), \\ \frac{dx_j}{dt} &= \sum_{k=1}^{n} a_{jk} x_k + F_j(\varphi_1, \ldots, \varphi_m) \quad (j = 1, \ldots, n), \end{aligned} \tag{2}$$

where a_{jk} are constants.

Assume that the functions Φ_i, F_j are continuous periodic functions of period 2π of the angular coordinates $\varphi_1, \ldots, \varphi_m$.

Theorem 3. *If all eigenvalues of the matrix $A = \{a_{jk}\}$ have negative real parts, then any solution of system (2) will be L-stable of class $(\varphi_1, \ldots, \varphi_m)$.*

Indeed, let us consider the auxiliary system

$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k \ (j = 1, \ldots, n). \tag{3}$$

Figure 4: Figure 4

By virtue of the well-known Lyapunov's theorem ([6], p. 35) there exists a positive definite quadratic form $v(x_1, \ldots, x_n)$, whose derivative, taken by virtue of the system (3), is equal to the function $w = -x_1^2 - \ldots - x_n^2$. Taking the derivative of the function v, by virtue of the system (2) we get

$$\frac{dv}{dt} = w + \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} F_j.$$

Consider now in space $\mathbb{R}$ the cylinder $x_1^2 + \ldots + x_n^2 = r^2$. As functions $\partial v/\partial x_j$ are linear with respect to the variables $x_1, \ldots, x_n$, and the functions F_j are bounded functions of arguments $\varphi_1, \ldots, \varphi_m$, then, having chosen r sufficiently large, we get on the surface of the cylinder and outside it the inequality $\dot{v} < -\varepsilon^2 < 0$. But this means that all trajectories of the system (2) fall with the growth of time inside the cylinder and remain these forever. Since the inner part of the cylinder goes over when folding $\mathbb{R}$ along all angular coordinates into a bounded set, we get, thus, in space $\mathbb{R}(\varphi_1, \ldots, \varphi_m)$ L-stability.

Corollary. Let there exist for the system (2) a continuously differentiable single-valued in space $\mathbb{R}$ function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$. Let us assume that the function v is periodic (with period 2π) in all angular coordinates $\varphi_1, \ldots, \varphi_m$, and the derivative of it $\dot{v}$, taken by virtue of the system (2), is sign-negative. If all eigenvalues of the matrix $A = \{a_{jk}\}$ have negative real parts, then any trajectory of the system (2) indefinitely approaches as $t \to \infty$ the invariant set lying on the set $\dot{v} = 0$.

In particular, if the set $\dot{v} = 0$ in space $\mathbb{R}(\varphi_1, \ldots, \varphi_m)$ contains as an invariant set only one point, then under the fulfillment of the conditions indicated above, we get unlimited approach of trajectories to uo to this point.

However, this particular case occurs very rarely in applications. More common is the case when there is a certain region of attraction of stable equilibrium positions, this region is bounded by separating surfaces passing through unstable equilibrium positions.

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Figure 5: Figure 5

Therefore, we limit ourselves to considering (3.11) in the range $0 \leqslant t \leqslant 4$. δ $\delta(x_1, x_2, 0) \neq 0, \ \pm\frac{\pi}{2}, \ \pm\pi, \ \pm\frac{3\pi}{2}$, then the maximum of the first periodic factor in (3.11) and the modulus of the maximum of the second periodic factor will occur at different t. Hence it is easy to deduce that the maximum $|u_1(x, 0, t)| < a_1$ over one period. Thus we arrive at the existence of points (x_1, x_2)- $x_2)$ outside the ellipse (3.9), which ensure (3.10) for all $0 \leqslant t \leqslant 4$, and consequently, for all $t \geqslant 0$.

If the function (3.11) were not periodic, which will be the case when ω/π is an irrational number, then we would not find points outside the ellipse (3.9) satisfying the requirement (3.10). In this case, the capture region with respect to x_1 would exactly coincide with the region brauned by the ellipse (3.9).

§ 4. INTEGRALS OF MOTION OF A SYSTEM OF FOUR EQUATIONS

Let the system of equations be given:

$$\frac{du_k}{dt} = \sum_{i=1}^{4} \varphi_{ki}(z+t)u_i \quad (k = 1, 2, 3, 4), \tag{4.1}$$

where $\varphi_{ki}(t)$ — are periodic real functions with period 1; z — parameter. The matrix of linearly independent solutions can be written in the form (1.2), where

$$W(z) = \left[\exp\left(-\int_0^z \Phi_{11}(\zeta)\,d\zeta\right);\ \exp\left(-\int_0^z \Phi_{22}(\zeta)\,d\zeta\right);\right.$$
$$\left.\exp\left(-\int_0^z \Phi_{33}(\zeta)\,d\zeta\right);\ \exp\left(-\int_0^z \Phi_{44}(\zeta)\,d\zeta\right)\right] w(z); \tag{4.2}$$

$$\Phi_{ik}(z) = \sum_{m=1}^{4} w_{im}(z)\varphi_{mk}(z); \tag{4.3}$$

the elements $w_{ik}(z)$ of the matrix $w(z)$ are periodic solutions of the nonlinear system

$$\frac{dw_{ik}}{dz} = w_{ik}\Phi_{ii} - \Phi_{ik}, \quad w_{ii} = 1 \quad (i, k = 1, 2, 3, 4). \tag{4.4}$$

System (4.4) decomposes into four independent systems of three equations in each (i — system number). If the solution of the system with number i is known, then it is easy to obtain the corresponding solution of the system with number m:

$$w_{mn} = \frac{w_{in}}{w_{im}}. \tag{4.5}$$

Therefore, it is sufficient to find 4 distinct periodic solutions for one any system (in [1] the existence of these periodic solutions is proved), so that then using (4.5) to find all functions necessary for constructing the matrix $w(z)$. Based on this, we will proceed below everywhere from the system with number $i = 2$.

Let us denote the first solution of this system by w_{21}, w_{22}, w_{23}, w_{24}, the second solution by v_{21}, v_{22}, v_{23}, v_{24}. Then, due to the reality of the coefficients $\Phi_{ik}(t)$, the other pair of solutions will be complex conjugate to the specified.

Figure 6: Figure 6

Let us denote further

$$\overline{\Phi}_{22} = \sum_{m=1}^{4} v_{2m}\,\varphi_{m2}. \tag{4.6}$$

Then the matrix $W(z)$ can be written as ($W(z)$ is defined up to a constant left multiplier, which is easy to see from (1.2)):

$$W(z) = \left[\exp\left(-\int_0^z \Phi_{22}^{*}\,d\zeta\right);\ \exp\left(-\int_0^z \Phi_{22}\,d\zeta\right);\ \exp\left(-\int_0^z \overline{\Phi}_{22}\,d\zeta\right);\ \exp\left(-\int_0^z \overline{\Phi}_{22}^{*}\,d\zeta\right)\right] \begin{pmatrix} w_{21}^{*} & 1 & w_{23}^{*} & w_{24}^{*} \\ w_{21} & 0 & w_{23} & w_{24} \\ v_{21} & 1 & v_{23} & v_{24} \\ v_{21}^{*} & 0 & v_{23}^{*} & v_{24}^{*} \end{pmatrix}. \tag{4.7}$$

Let us separate the real and imaginary parts in the functions w_{lk} and v_{lk}:

$$w_{lk} = \overline{w}_{lk} + i\widetilde{w}_{lk}, \quad v_{lk} = \overline{v}_{lk} + i\widetilde{v}_{lk}. \tag{4.8}$$

We will assume that the following determinant is non-zero:

$$d(z) = \begin{vmatrix} \overline{w}_{21} & 1 & \overline{w}_{23} & \overline{w}_{24} \\ \widetilde{w}_{21} & 0 & \widetilde{w}_{23} & \widetilde{w}_{24} \\ \overline{v}_{21} & 1 & \overline{v}_{23} & \overline{v}_{24} \\ \widetilde{v}_{21} & 0 & \widetilde{v}_{23} & \widetilde{v}_{24} \end{vmatrix} \neq 0. \tag{4.9}$$

Let us pass from the variables $u_1 = u_1(x,z,t)$, $u_2 = u_2(x,z,t)$, $u_3 = u_3(x,z,t)$, $u_4 = u_4(x,z,t)$ to the variables

$$\begin{aligned} s_1 &\equiv s_1(x,z,t) = \overline{w}_{21}(z+t)u_1 + u_2 + \\ &\quad + \overline{w}_{23}(z+t)u_3 + \overline{w}_{24}(z+t)u_4, \\ s_2 &\equiv s_2(x,z,t) = \widetilde{w}_{21}(z+t)u_1 + \widetilde{w}_{23}(z+t)u_3 + \widetilde{w}_{24}(z+t)u_4, \\ s_3 &\equiv s_3(x,z,t) = \overline{v}_{21}(z+t)u_1 + u_2 + \\ &\quad + \overline{v}_{23}(z+t)u_3 + \overline{v}_{24}(z+t)u_4, \\ s_4 &\equiv s_4(x,z,t) = \widetilde{v}_{21}(z+t)u_1 + \widetilde{v}_{23}(z+t)u_3 + \widetilde{v}_{24}(z+t)u_4. \end{aligned} \tag{4.10}$$

Denosting for brevity

$$s_1(x,z,0) \equiv \xi_1, \quad s_2(x,z,0) \equiv \xi_2, \quad s_3(x,z,0) \equiv \xi_3, \quad s_4(x,z,0) \equiv \xi_4, \tag{4.11}$$

we bring equation (1.12) to the form

$$\left[\exp\left(-\int_z^{z+t} \Phi_{22}^{*}\,d\zeta\right);\ \exp\left(-\int_z^{z+t} \Phi_{22}\,d\zeta\right);\ \exp\left(-\int_z^{z+t} \overline{\Phi}_{22}\,d\zeta\right);\ \exp\left(\int_z^{z+t} \Phi_{22}^{*}\,d\zeta\right)\right] \begin{pmatrix} s_1 - is_2 \\ s_1 + is_2 \\ s_3 + is_4 \\ s_3 - is_4 \end{pmatrix} = \begin{pmatrix} \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 \\ \xi_3 + i\xi_4 \\ \xi_3 - i\xi_4 \end{pmatrix}. \tag{4.12}$$

Figure 7: Figure 7

which is equivalent to the following two equations:

$$(s_1 + is_2) \exp\left(-\int_z^{z+t} \Phi_{22}\, d\xi\right) = \bar{\xi}_1 + i\bar{\xi}_2, \tag{4.13}$$

$$(s_3 + is_4) \exp\left(-\int_z^{z+t} \bar{\Phi}_{22}\, d\xi\right) = \bar{\xi}_3 + i\bar{\xi}_1. \tag{4.14}$$

Let

$$\Phi_{22} = \eta(z) + i\,\omega(z), \quad \bar{\Phi}_{22} = \bar{\eta}(z) + i\,\bar{\omega}(z). \tag{4.15}$$

Let us assume that

$$\int_0^1 \eta(\xi)\, d\xi < 0, \quad \int_0^1 \bar{\eta}(\xi)\, d\xi < 0, \tag{4.16}$$

$$\int_0^1 \omega(\xi)\, d\xi \neq 0, \quad \int_0^1 \bar{\omega}(\xi)\, d\xi \neq 0. \tag{4.17}$$

Introduce notations:

$$\xi_1^2 + \xi_2^2 = Q^2(x, z) = Q^2, \tag{4.18}$$

$$\xi_4^2 + \xi_4^2 = \bar{Q}^2(x, z) = \bar{Q}^2, \tag{4.19}$$

$$\delta(x, z, t) = \arg(s_1 + is_2), \tag{4.20}$$

$$\bar{\delta}(x, z, t) = \arg(s_s + is_1). \tag{4.21}$$

Then equations (4.13), (4.14) can be written in the form

$$s_1^2 + s_2^2 = Q^2 \exp 2 \int_2^{z+t} \eta\, d\xi, \tag{4.22}$$

$$s_3^3 + s_3^2 = \bar{Q}^2 \exp 2 \int_2^{z+t} \bar{\eta}\, d\xi, \tag{4.23}$$

$$\delta(x, z, t) = \int_2^{z+t} \omega\, d\xi + \delta(x, z, 0), \tag{4.24}$$

$$\bar{\delta}(x, z, t) = \int_2^{z+t} \bar{\omega}\, d\xi + \bar{\delta}(x, z, 0). \tag{4.25}$$

Let us cas call formylas (4.22), (4.23) the first integrals, and formylas (4.24), (4.25) – the second integrals of system (4.1).

§ 5. Investigation of first integrals of system (4.1)

Soluue equations (4.10) for us equations u_1, u_2, u_3, u_4, which is possible by virtue of (4.9):

$$u_i = a_{11}(z+t)\, s_1 + a_{12}(z+t)\, s_1 + a_{13}(z+t)\, s_3 + a_{14}(z+t)\, s_4 \tag{5.1}$$
$$(i = 1, 2, 3, 4).$$

Figure 8: Figure 8

where $a_{ik}(z+t)$ is easily expressed in terms of the system coefficients (4.10).

Let us assign to z and t certain values, and the components of vector $\bar{\mathbf{x}}$ will be chosen such that Q and $\bar{Q}$ have a constant value. Then the values s_1, s_2 will be coordinates of points of the circumference (4.18), and s_3, s_4 — coordinates of poists of the circumference (4.19). It is required to find the maximum of $|u_i|$, assuming that s_1, s_2, s_3, s_4 are subject to two constraint equations (4.22) and (4.23). For this, it is necessary to solve the conditional extremum problem [2]. Denotiing by $m_i(\mathbf{x}, z, t)$ the maximum of the value $|u_i|$, we obtain

$$m_i(\mathbf{x}, z, t) = Q(\mathbf{x}, z)\sqrt{a_{i1}^2(z+t)+a_{i2}^2(z+t)}\exp\left(\int_z^{z+t}\eta\,d\xi\right) + \bar{Q}(\mathbf{x}, z)\sqrt{a_{i3}^2(z+t)+a_{i3}^2(z+t)}\exp\left(\int_z^{z+t}\bar{\eta}\,d\xi\right). \qquad (5.2)$$

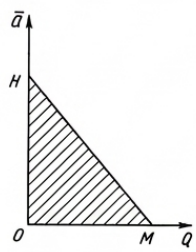

Fig. 1

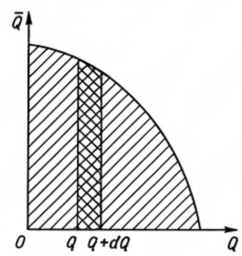

Fig. 2

Let us find on the plone of variables Q, $\bar{Q}$ such a region, that coordinates of points $(Q, \bar{Q})$ from this region, substituted into (5.2), gave the inequality (2.5) for all t from the interval (2.6) or, what is the same, from the interval (2.7) (with the assumption that $z = \text{const}$).

Let us preliminarily indicate such a region, amony point $(Q, \bar{Q})$ of which leads to the inequality (2.5) only for a certain value t from the interval (2.7). This region will be orgrnded by the ptraight line

$$Q\sqrt{a_{i1}^2(z+t)+a_{i2}^2(z+t)}\exp\int_z^{z+t}\eta\,d\xi + \bar{Q}\sqrt{a_{i3}^2(z+t)+a_{i3}^2(z+t)}\exp\int_z^{z+t}\bar{\eta}\,d\xi = a_i \qquad (5.3)$$

and coordinate aces (Fig. 1). On the coordinate oxes, this line cuts off segsents

$$OM = \frac{a_i\exp\left(-\int_z^{z+t}\eta\,d\xi\right)}{\sqrt{a_{i1}^2+a_{i3}^2}}, \qquad ON = \frac{a_i\exp\left(-\int_z^{z+t}\bar{\eta}\,d\xi\right)}{\sqrt{a_{i3}^2+a_{i4}^2}}. \qquad (5.4)$$

Figure 9: Figure 9

For various t, the line NM will be positioned differently, therefore, for each t there will be its own region OMN. If we now take the region which is the intersection of regions for various t from the interval (2.7), we obtain a region, every point of which, when substituted into (5.2), leads (5.2), leads to the inequality (2.5) for all t from the interval (2.7). This region will be located in the first quadrant, bounded by the coordinate axes and some curve (Fig. 2), the parametric equation of which can be obtained as follows.

Let the point $(Q_0, \widetilde{Q}_0)$ be the point of intersection of the line

$$\widetilde{Q} = xQ, \tag{5.5}$$

where $0 \leqslant x < \infty$, with the line (5.3). Obviously

$$Q_0 = \frac{a_i}{\sqrt{a_{i1}^2 + a_{i2}^2}\exp\int\limits_z^{z+t}\eta\,d\zeta + x\sqrt{a_{i3}^2 + a_{i4}^2}\exp\int\limits_z^{z+t}\overline{\eta}\,d\zeta},$$
$$\widetilde{Q}_0 = \frac{a_i\,x}{\sqrt{a_{i1}^2 + a_{i2}^2}\exp\int\limits_z^{z+t}\eta\,d\zeta + x\sqrt{a_{i3}^2 + a_{i4}^2}\exp\int\limits_z^{z+t}\overline{\eta}\,d\zeta}. \tag{5.6}$$

Let us denote by $M_i(x, z)$ the minimum of Q_0 in the interval (2.7). Then the parametric equation of the boundary curve will have the form (x is a parameter)

$$Q = M_i(x, z), \quad \widetilde{Q} = x\,M_i(x, z). \tag{5.7}$$

Eliminating from (5.7) the parameter x, we obtain

$$\widetilde{Q} = f_i(Q, z). \tag{5.8}$$

Let us now find the 4-dimensional volume, the coordinates of the points (s_1, s_2, s_3, s_4) of which, substituted into (4.23) will gives Q, $\widetilde{Q}$, coorserstryying to the coordinates of the tower of the shaded area in Fig. 2.

Let us fix some Q. Then the values Q the $\widetilde{Q}$ will be contained within the limits

$$0 \leqslant \widetilde{Q} \leqslant f_i(Q, z). \tag{5.9}$$

To these values of Q and $\widetilde{Q}$ correspond points (s_1, s_2), leying on the circumference (4.22), and points (s_3, s_4) — the interior of the circle (4.23), the area of which is pqual

$$\pi f_i^2(Q, z)\exp 2\int\limits_z^{z+t}\overline{\eta}\,d\zeta. \tag{5.10}$$

This valuees Q, $\widetilde{Q}$, cooteerstryying to the shaded strip in Fig. 2, correnspond the coordinate s_1, s_2 of the tower of the ring, the area of which is paval to

$$2\pi Q\exp 2\int\limits_z^{z+t}\eta\,d\zeta\,dQ. \tag{5.11}$$

The 4-dimensional volume, coorterstryying to the valuees Q, $\widetilde{Q}$ from the shaded strip in Fig. 2, will will be pauul to the product (5.10) and (5.11):

$$2\pi^2\exp\Big(2\int\limits_z^{z+t}(\eta + \overline{\eta})\,d\zeta\Big)f_i^2(Q, z)\,Q\,dQ. \tag{5.12}$$

Figure 10: Figure 10

If we now integrate (5.12) in the range from zero to the value $Q_i(z)$, equal to the root of the equation

$$f_i(Q_i, z) = 0, \tag{5.13}$$

then we will obtain the phase volume $V_{i0}(z, t)$, the coordinates of the toinks (s_1, s_2, s_3, s_4) of which, substituted in (4.22) and (4.23), will give the value б Q, $\overline{Q}$, corrtersctwing to the hatched area in Fig. 2:

$$V_{i0}(z, t) = 2\pi^2 \exp\left(2\int\limits_z^{z+t} (\eta + \overline{\eta})\, d\zeta\right) \int\limits_0^{Q_i(z)} f_i^2(Q, z) Q dQ. \tag{5.14}$$

Now it is not difficult to find the 4-dimensional oblame in the prostprance u_1, u_2, u_3, u_4, the coordinates of the toinks of which, substituted first in (4.10), and let in (4.22) and (4.23), will give the points $(Q, \overline{Q})$, leying in the hatched area in Fig. 2. For this it is necessary to disdile (5.14) ha $d(z+t)$ (see (4.9)). The formyla holds [1]

$$d(z+t) = d(z) \exp \int\limits_z^{z+t} \left(2\eta + 2\overline{\eta} - \sum_{i=1}^{4} \phi_{ii}(\zeta)\right) d\zeta. \tag{5.15}$$

Dividing (5.14) ha (5.15), we holyvin

$$V_i(z, t) = \frac{2\pi^2}{d(z)} \exp\left(\int\limits_z^{z+t} \sum_{i=1}^{4} \phi_{ii}(\zeta)\, d\zeta\right) \int\limits_0^{Q_i(z)} f_i^2(Q, z) Q dQ. \tag{5.16}$$

Assuming in (5.16) $t = 0$, we holyain the obleme

$$V_i(z, 0) = \frac{2\pi^2}{d(z)} \int\limits_0^{Q_i(z)} f_i^2(Q, z) Q dQ, \tag{5.17}$$

which will lighit from bnow the measy of caxture with recet to the koopdinate x_i.

In the cayae, ɪɪnen it is required to find the huwrer granuny meases of the caxture with resteecet to neevolisan nepeмenles $x_i, x_{i+1}, \ldots$, we monst on the plockote Q, $\overline{Q}$ firstɯdly find the granuisy nepeceverion obliacch, organuned us kurivsans $f_i(Q, z)$, $f_{i+1}(Q, z)$, ..., then find the granıuty curuiʋy and nedstabite it in (5.13) and (5.17) instecto $f_i(Q, z)$.

In conslosion, I express my gratitude to A. D. Myshkis and Yu. S. Bogdanov for the valable discussion of the work.

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Figure 11: Figure 11

$$A(t) = [t^{-1},\ 1]\,S \begin{vmatrix} t^m, & 0 \\ -\sum_{k=1}^{m-1} \bar{q}_{m-1}\, t^k, & 1 \end{vmatrix} \bar{Z}^{-1}(t)\, BX_2 S^{-1}[t,\ 1], \tag{4.66}$$

where X_2 is found by formyla (4.63), and $\bar{Z}(t)$ is the same as in (4.65).

Let us expand $A(t)$ in powers of t, find the coeffficients for positive powers of t and equate them to zero. From this we holyain, tho (10) holds, its if $B = \begin{bmatrix} b, & 0 \\ b_{21}, & b \end{bmatrix}$, and $D(A) = D(B) = b^2$, and for $b \neq 0$ we have a non-singular case.

Let $\tilde{h}_{21}^{(1)}(t) \neq 0$. Then, tion, using [7], we have

$$Y_2 = \begin{vmatrix} 1, & 0 \\ \ln t, & 1 \end{vmatrix} \bar{Z}(t)\,[\tilde{h}_{21}^{(1)},\ 1] \begin{vmatrix} t^m, & 0 \\ \bar{q}(t), & 1 \end{vmatrix}, \tag{4.67}$$

where $\bar{Z}(t)$ and $\bar{q}(t)$ are the same as in (4.65). From this

$$A(t) = [t^{-1},\ 1]\,S \begin{vmatrix} t^m, & 0 \\ -\sum_{k=1}^{m-1} \bar{q}_{m-k}\, t^k, & 1 \end{vmatrix} [(\tilde{h}_{21}^{(1)})^{-1},\ 1] \times$$

$$\times \bar{Z}^{-1}(t) \begin{vmatrix} 1, & 0 \\ -\ln t, & 1 \end{vmatrix} BX_S S^{-1}[t,\ 1], \tag{4.68}$$

where $\bar{Z}(t)$ is the same as in (4.67), and X_2 is found by formyla (4.63). Here we have (10) if $B = \begin{bmatrix} b\tilde{h}_{21}^{(1)}(t), & 0 \\ b_{21} & b \end{bmatrix}$, and $D(A) = D(B)\left(\tilde{h}_{21}^{(1)}(t)\right)^{-1} = b^2$, for $b \neq 0$ we have a non-singular case.

If $n \geqslant m+2$, then $\tilde{h}_{21}^{(1)}(t) = \tilde{h}_{21}^{(1)}(t)$, and we have the same result as and for $1 < n < m+2$, $\tilde{h}_{21}^{(1)} = 0$.

g) $m > 1$, $\tilde{h}_{21}^{(1)} \neq 0$. According to (2.6) [7], we have

$$X_2 = \begin{vmatrix} 1, & 0 \\ \ln t, & 1 \end{vmatrix} \left([0,\ 1] + \sum_{k=0}^{m-1} \begin{vmatrix} 0, & 0 \\ d_{21}^{(0)}, & d_{22}^{(0)} \end{vmatrix} t^{-k} + \right.$$

$$\left. + \begin{vmatrix} \tilde{h}_{21}^{(1)}, & 0 \\ d_{21}^{(m)}, & d_{22}^{(m)} \end{vmatrix} t^{-m} + \sum_{k=m+1}^{\infty} D_k{}^{k-k} \right). \tag{4.69}$$

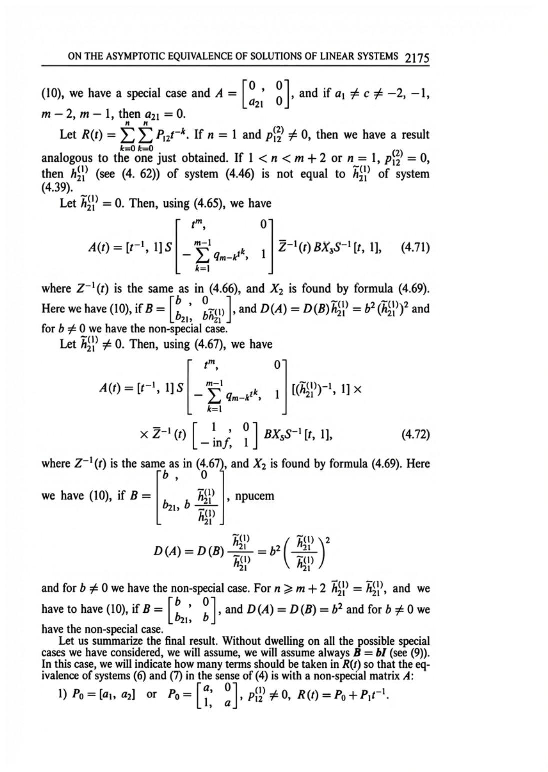
Let $R(t) = P_0$, then

$$A(t) = \begin{vmatrix} 1, & 0 \\ -t, & 1 \end{vmatrix} BX_S S^{-1}[t,\ 1]\, t^e, \tag{4.70}$$

where X_2 is found by formyla (4.69).

Let us expand $A(t)$ in powers of t and calculate the coefficients (they will contain $\ln t$). Let us equate these coefficients and the terms of the constant matrix containing $\ln t$, to zero. From this we obtain that always, when is fulfilled

Figure 12: Figure 12



(10), we have a special case and $A = \begin{bmatrix} 0\ , & 0 \\ a_{21} & 0 \end{bmatrix}$, and if $a_1 \neq c \neq -2, -1, m-2, m-1$, then $a_{21} = 0$.

Let $R(t) = \sum_{k=0}^{n} \sum_{k=0}^{n} P_{12} t^{-k}$. If $n = 1$ and $p_{12}^{(2)} \neq 0$, then we have a result analogous to the one just obtained. If $1 < n < m+2$ or $n = 1$, $p_{12}^{(2)} = 0$, then $h_{21}^{(1)}$ (see (4. 62)) of system (4.46) is not equal to $\widetilde{h}_{21}^{(1)}$ of system (4.39).

Let $\widetilde{h}_{21}^{(1)} = 0$. Then, using (4.65), we have

$$A(t) = [t^{-1},\ 1]\, S \begin{bmatrix} t^m, & 0 \\ -\sum_{k=1}^{m-1} q_{m-k} t^k, & 1 \end{bmatrix} \overline{Z}^{-1}(t)\, B X_3 S^{-1} [t,\ 1], \qquad (4.71)$$

where $Z^{-1}(t)$ is the same as in (4.66), and X_2 is found by formula (4.69). Here we have (10), if $B = \begin{bmatrix} b\ , & 0 \\ b_{21}, & b\widetilde{h}_{21}^{(1)} \end{bmatrix}$, and $D(A) = D(B)\widetilde{h}_{21}^{(1)} = b^2 (\widetilde{h}_{21}^{(1)})^2$ and for $b \neq 0$ we have the non-special case.

Let $\widetilde{h}_{21}^{(1)} \neq 0$. Then, using (4.67), we have

$$A(t) = [t^{-1},\ 1]\, S \begin{bmatrix} t^m, & 0 \\ -\sum_{k=1}^{m-1} q_{m-k} t^k, & 1 \end{bmatrix} [(\widetilde{h}_{21}^{(1)})^{-1},\ 1] \times$$

$$\times\, \overline{Z}^{-1}(t) \begin{bmatrix} 1\ , & 0 \\ -\operatorname{in} f, & 1 \end{bmatrix} B X_3 S^{-1} [t,\ 1], \qquad (4.72)$$

where $Z^{-1}(t)$ is the same as in (4.67), and X_2 is found by formula (4.69). Here we have (10), if $B = \begin{bmatrix} b\ , & 0 \\ b_{21}, & b\,\dfrac{\widetilde{h}_{21}^{(1)}}{\widetilde{h}_{21}^{(1)}} \end{bmatrix}$, npucem

$$D(A) = D(B)\,\frac{\widetilde{h}_{21}^{(1)}}{\widetilde{h}_{21}^{(1)}} = b^2 \left(\frac{\widetilde{h}_{21}^{(1)}}{\widetilde{h}_{21}^{(1)}} \right)^2$$

and for $b \neq 0$ we have the non-special case. For $n \geqslant m+2$ $\widetilde{h}_{21}^{(1)} = \widetilde{h}_{21}^{(1)}$, and we have to have (10), if $B = \begin{bmatrix} b\ , & 0 \\ b_{21}, & b \end{bmatrix}$, and $D(A) = D(B) = b^2$ and for $b \neq 0$ we have the non-special case.

Let us summarize the final result. Without dwelling on all the possible special cases we have considered, we will assume, we will assume always $\boldsymbol{B} = \boldsymbol{bI}$ (see (9)). In this case, we will indicate how many terms should be taken in $R(t)$ so that the equivalence of systems (6) and (7) in the sense of (4) is with a non-special matrix A:

1) $P_0 = [a_1,\ a_2]$ or $P_0 = \begin{bmatrix} a, & 0 \\ 1, & a \end{bmatrix}$, $p_{12}^{(1)} \neq 0$, $R(t) = P_0 + P_1 t^{-1}$.

Figure 13: Figure 13