PROOF OF THE NONISOMORPHISM OF THE SPACES OF SMOOTH FUNCTIONS ON AN INTERVAL AND ON A SQUARE

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Abstract

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PROOF OF THE NONISOMORPHISM OF THE SPACES OF SMOOTH FUNCTIONS ON AN INTERVAL AND ON A SQUARE

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Let I^n be the n-dimensional unit cube in n-dimensional Euclidean space. Denote by $C^{(s)}(I^n)$ the space of all s times continuously differentiable functions defined on the cube I^n, with norm

$$\|f(x)\|_{C^s(I^n)} = \sum_{k_1+k_2+\ldots+k_n \le s} \sup_{x\in I^n} \left| \frac{\partial^{k_1+k_2+\ldots+k_n} f(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}} \right|.$$

The purpose of the present note is to outline a proof of the following theorem.

Theorem 1. *For any integers $p \ge 0$, $s \ge 1$, $n \ge 2$, the space $C^{(s)}(I^n)$ is not isomorphic to the space $C^{(p)}(I)$, i.e., between the elements of these spaces there exists no one-to-one continuous and linear correspondence.*

Let M be a closed linear subspace in a Banach space B. We shall call M **weakly complemented** in B if there exists a continuous linear extension operation $E : M^* \to B^*$ of linear functionals from the subspace M to the whole space B. We shall call M **complemented** in B if there exists a linear continuous projection operator $P : B \to M$ of the space B onto the subspace M. If the subspace M is complemented in B, then it is also weakly complemented in B, since one may take the adjoint operator P^* as the extension operation. The converse, generally speaking, is false.

A. A. Milyutin had the idea of obtaining Theorem 1 as a consequence of the following, formulated by him as a conjecture, more general Theorem 2. Essentially due to him as well is the result important for the proof of Theorem 2—Theorem 3.

Theorem 2. *For any integers $s \ge 1$, $n \ge 2$, and any compact Hausdorff space K, the space $C^{(s)}(I^n)$ is not isomorphic to any weakly complemented subspace of the space $C(K)$ of continuous functions on K.*

Let us note that from Theorem 2 there follows the following proposition, formulated without proof by A. Grothendieck in the paper ([1]):

The space $C^{(s)}(I^n)$ $(s \geq 1, n \geq 2)$ is not isomorphic to a complemented subspace of the space of continuous functions on a suitable compactum.

We shall say that a Banach space M has the **E-property** if M is weakly complemented in every space $B \supset M$. We say that a Banach space M has the **P-property** if M is complemented in every space $B \supset M$.

The following theorem is a combination of well-known ([2]) theorems:

* It is interesting to compare Theorem 1 with the theorem of A. A. Milyutin ([5]), asserting, in particular, that the spaces of continuous functions on cubes of different numbers of dimensions are isomorphic to one another.

Lemma 1. In order that the space M have the P-property, it is necessary and sufficient that M be isomorphic to some complemented subspace of the space $C^{**}(K)$.

Here $C^{**}(K)$ denotes the second conjugate of the space of continuous functions on some compact Hausdorff space K.

Theorem 3*. In order that the space M have the E-property, it is necessary and sufficient that M be isomorphic to some weakly complemented subspace of the space $C(K)$ of continuous functions on some compact Hausdorff space K.

Proof. If M is isomorphic to some weakly complemented subspace of the space $C(K)$, then the first conjugate M^* is isomorphic to some complemented subspace of the space $C^*(K)$, and consequently the second conjugate M^{**} is isomorphic to a complemented subspace of the space $C^{**}(K)$. By Lemma 1, the space M^{**} has the P-property. Let $I_M : M \to M^{**}$ be the operator of the natural isometric embedding of the space M into the space M^{**}. Let $J : M \to B$ be the operator of the identity embedding of the space M into some space $B \supset M$. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{I_M} & M^{**} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J^{**}} \\ B & \xrightarrow{I_B} & B^{**} \end{array}$$

Since every Banach space is weakly complemented in its second conjugate, we obtain from the diagram that the space M is weakly complemented in the space B^{**}. It also follows from the diagram that $M \subset B \subset B^{**}$. Consequently, the space M is weakly complemented in the space $B \supset M$. The theorem is proved.

For simplicity of exposition, in proving Theorem 2 we restrict ourselves to the consideration of the space $C^{(1)}(I^n)$, i.e., the case when $s = 1$. Denote by $C^{(1)}(S^n)$ the space of continuously differentiable functions on the n-dimensional unit sphere S^n, and by $C(S^n)$ the space of continuous tangent vector fields on the sphere S^n.

Lemma 2. The space $C^{(1)}(S^n)$ is isomorphic to the space $C^{(1)}(I^n)$.

For what follows it is more convenient to deal not with the space $C^{(1)}(I^n)$, but with the space isomorphic to it, $C_0^{(1)}(S^n)$, of those continuously differentiable functions on the sphere S^n whose integral over the surface of the sphere is equal to zero. Let

$$G : C_0^{(1)}(S^n) \to C(S^n)$$

be the isomorphic embedding of the space $C_0^{(1)}(S^n)$ into the space $C(S^n)$, effected by the formula $\operatorname{grad} f(s) = g(s)$, where $f(s) \in C_0^{(1)}(S^n)$, and $g(s)$ is the gradient of the function $f(s)$ at the point $s \in S^n$. Then the operator $G^* : [C(S^n)]^* \to [C_0^{(1)}(S^n)]^*$ maps the space $[C(S^n)]^*$, conjugate to the space $C(S^n)$, onto the space $[C_0^{(1)}(S^n)]^*$.

By ψ_s we denote the linear functional in the space $C_0^{(1)}(S^n)$ defined by the formula $\psi_s(f) = f(s)$, where $f \in C_0^{(1)}(S^n)$, $s \in S^n$. It is clear that the correspondence $s \to \psi_s$ effects a mapping satisfying a Lipschitz condition from the sphere S^n into the space of linear functionals $[C_0^{(1)}(S^n)]^*$. Denote by $U_n = \{u\}$ the group of all isometric transformations of the sphere S^n onto itself.

* Theorem 3 is a modification of an unpublished result of A. A. Miljutin. The author expresses deep gratitude to A. A. Miljutin for the opportunity to become acquainted with this result, as well as for useful discussions.

Lemma 3. Let $s \to \varphi_s$ be a mapping of the sphere S^n into the space $[C(S^n)]^*$, satisfying the conditions

(a) $G^*\varphi_s = \psi_s$ for every point $s \in S^n$;

(b) $\|\varphi_{s_1} - \varphi_{s_2}\|_{[C(S^n)]^*} \le \omega(|s_1 - s_2|)$,

where $|s_1 - s_2|$ is the distance between the points s_1 and s_2 of the sphere S^n, and $\omega(\delta)$ is a certain function of the variable δ, with $\omega(\delta) \to 0$ as $\delta \to 0$.

Then there exists a mapping $s \to \widetilde{\varphi}_s$ of the sphere S^n into the space $[C(S^n)]^*$, satisfying conditions (a), (b) and the condition:

(c) for every point $s \in S^n$, every $g \in C(S^n)$, and every element $u \in U_n$,

$$\widetilde{\varphi}_{us}(g(t)) = \widetilde{\varphi}_s(g(ut)),$$

where by us is denoted the image of the point s under the isometric transformation u of the sphere S^n onto itself.

The proof of the lemma is based on the fact that, as the functional $\widetilde{\varphi}_s$, one may take the functional defined by the formula

$$\widetilde{\varphi}_s(g) = \int_{U_n} \varphi_{us}(g(u^{-1}t))\, \mu(du),$$

where $g \in C(S^n)$, and the integration is with respect to the Haar measure μ on the compact group U_n, where it is assumed that $\mu(U_n) = 1$.

Lemma 4. The mapping $s \to \widetilde{\varphi}_s$ of the sphere S^n into the space $[C(S^n)]^*$, satisfying conditions (a) and (c) of Lemma 3, is unique; moreover, the functional $\widetilde{\varphi}_s$ can be represented by the following integral over the surface of the sphere S^n:

$$\widetilde{\varphi}_s(g) = \frac{1}{\sigma(S^n)} \int_{S^n} \frac{A(|t-s|)}{(\sin|t-s|)^{n-1}}\, g_s(t)\, \sigma(dt),$$

where $g_s(t)$ denotes the projection of the vector $g(t)$ onto the unit vector tangent to the sphere S^n at the point t and directed toward the point s;

$$A(|t-s|) = \int_{|t-s|}^{\pi} \sin^{n-1}\theta\, d\theta; \qquad \sigma(S^n) \text{ is the surface area of the sphere } S^n.$$

Proof. If the mapping $s \to \widetilde{\varphi}_s$ of the sphere S^n into the space $[C(S^n)]^*$ satisfies condition (c) of Lemma 3, then from this it necessarily follows that the functional φ_s can be represented in the form

$$\widetilde{\varphi}_s(g(t)) = \int_{S^n} B(|t-s|)\, g_s(t)\, \sigma(dt),$$

where $B(|t-s|)$ is a generalized function of measure type in the argument $|t-s|$.

Let us emphasize that here we have essentially used the fact that the group U_n is the group of all isometric transformations of the sphere S^n. The requirement contained in condition (a) of Lemma 3, that $G^*\varphi_s = \psi_s$, means exactly the following:

$$\widetilde{\varphi}_s(\operatorname{grad} f(t)) = f(s) \quad \text{for every function } f(t) \in C_0^{(1)}(S^n).$$

A simple computation shows that, in order that the last requirement be satisfied, the kernel $B(|t-s|)$ must have the following special form:

$$B(|t-s|) = \frac{1}{\sigma(S^n)} \frac{A(|t-s|)}{(\sin|t-s|)^{n-1}}.$$

Lemma 5. Let $s \to \widetilde{\varphi}_s$ be a mapping of the sphere S^n into the space $[C(S^n)]^*$, satisfying conditions (a) and (c) of Lemma 3. Then for any points $s_1, s_2 \in S^n$,

$$\|\widetilde{\varphi}_{s_1} - \widetilde{\varphi}_{s_2}\|_{[C(S^n)]^*} \geq L\,|s_1 - s_2|\, \ln\frac{1}{|s_1 - s_2|},$$

where L is a constant independent of the points s_1 and s_2 of the sphere S^n.

Lemma 5 is proved on the basis of a special representation obtained for the functionals φ_s in Lemma 4.

From Lemma 3 and Lemma 5 it follows that

Theorem 4. *Whatever the constant $N > 0$, there does not exist a mapping $s \to \varphi_s$ of the sphere S^n into the space $[C(S^n)]^*$ satisfying the conditions*

a) *for every point $s \in S^n$, $G^*\varphi_s = \psi_s$;*

b) *for any two points $s_1, s_2 \in S^n$,*

$$\|\varphi_{s_1} - \varphi_{s_2}\|_{[C(S^n)]^*} \leqslant N\|\psi_{s_1} - \psi_{s_2}\|_{[C_0^{(s)}(S^n)]^*}.$$

It follows from Theorem 4 that the operator $G : C_0^{(1)}(S^n) \to C(S^n)$ realizes an isomorphism of the space $C_0^{(1)}(S^n)$ with a subspace of the space $C(S^n)$ having no weak complement. Hence, from Theorem 3 and Lemma 2, Theorem 2 follows for the case of the space $C^{(1)}(I^n)$.

Let us note that, by the Bartle-Graves theorem ([3]), for any linear continuous mapping $T : E \to F$ of a Banach space E onto a Banach space F, and any continuous mapping $t \to \psi_t$ of a paracompact K into the space F, there exists a continuous mapping $t \to \varphi_t$ of the paracompact K into the space E such that $T\varphi_t = \psi_t$ for every $t \in K$. Thus Theorem 4 indicates the impossibility of generalizing the Bartle-Graves theorem, for example, to the case of continuously differentiable mappings of the sphere S^n into the spaces F and E, respectively. It is interesting to compare these assertions with certain results of A. Grothendieck ([4]).

Finally, let us consider the space $L_\infty^{(s)}(I^n)$ of functions defined on the cube I^n, having all generalized derivatives in the sense of Sobolev up to order s inclusive, each of these derivatives being a measurable and almost everywhere bounded function. The norm in this space, as is known, is defined by the equality

$$\|f(x)\|_{L_\infty^{(s)}(I^n)} = \sum_{k_1+k_2+\cdots+k_n \leq s} \operatorname*{vrai\,sup}_{x \in I^n} \left| \frac{\partial^{k_1+k_2+\cdots+k_n} f(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} \right|.$$

Theorem 5. *For any integers $s \geqslant 1$ and $n \geqslant 2$, and any compact Hausdorff space K, the space $L_\infty^{(s)}(I^n)$ is not isomorphic to any weakly complemented subspace of the space $C(K)$; in particular, the space $L_\infty^{(s)}(I^n)$ is not isomorphic to any weakly complemented subspace of the space $L_\infty^{(s)}(I)$.*

Theorem 5 follows from Theorem 2, since the space $C^{(s)}(I^n)$ is weakly complemented in the space $L_\infty^{(s)}(I^n) \supset C^s(I^n)$.

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Note: Figure translations are in progress. See original paper for figures.