Conditions for the existence of bounded solutions of a difference equation in a Banach space

Authors: V. S. Kim

Date: 1967-01-01T00:00:00+00:00

Abstract

We consider the difference equation

$$x(m+1) = A(m)x(m) + u(m), \tag{1}$$

where $x(m)$, $u(m)$ are functions of the argument m $(m = 0, 1, 2, ...)$, the values of which belong to some Banach space E; $A(m)$ is a linear operator-function mapping (for each m) the space E into itself. In this article, conditions are derived under which the estimate

$$\|y(m)\|_E \leq N \exp[-\alpha(m - m_0)] \|y(m_0)\|_E,$$

holds for any solution of equation $y(m+1) = A(m)y(m)$, where m_0 is a fixed natural number; $N > 0$ and $\alpha > 0$ are constants independent of m_0. It is proved that this inequality and a certain additional condition (for $u(m)$) guarantee the boundedness of any solution to equation (1). The case where $A(m)$ does not depend on m is also investigated. Bibliography: 5 items.

Full Text

Preamble

This section investigates the stability and asymptotic behavior of linear discrete-time systems. We consider the non-homogeneous difference equation:

$$x(m+1) = A(m)x(m) + u(m), \quad m = 0, 1, 2, ... \tag{1}$$

where $x(m)$ is the state vector in a Banach space E, $A(m)$ is a linear bounded operator acting on E, and $u(m)$ is an external input sequence. The corresponding homogeneous equation is given by:

$$y(m+1) = A(m)y(m) \tag{2}$$

The solution to the homogeneous equation can be expressed using the evolution operator $W(m,s)$, defined for $m \geq s$ as $W(m,s) = A(m-1)A(m-2)\dots A(s)$, with $W(s,s) = I$. It follows that $W(m+1,s) = A(m)W(m,s)$. The norm of the evolution operator satisfies the inequality:

$$\|W(m,s)\| \leq \prod_{k=s}^{m-1} \|A(k)\| \tag{3}$$

We define the sequence spaces l_p $(1 \leq p < \infty)$ and m (the space of bounded sequences). A sequence $x = \{x(k)\}_{k=0}^{\infty}$ belongs to l_p if $\|x\|_{l_p} = (\sum \|x(k)\|^p)^{1/p} < \infty$, and belongs to m if $\|x\|_m = \sup \|x(k)\| < \infty$.

1. Stability Estimates for Homogeneous Systems

We establish conditions under which the solutions of the homogeneous system (2) exhibit exponential decay. Suppose that for any input sequence $u \in l_p$, the solution $x(m)$ of the non-homogeneous system (1) with initial condition $x(m_0) = 0$ remains bounded. Then, there exist constants $N(m_0) > 0$ and $\alpha > 0$ such that the solution $y(m)$ of (2) satisfies:

$$\|y(m)\| \leq N(m_0)\exp[-\alpha(m-m_0)^{1/p}]\|y(m_0)\|, \quad m > m_0 \tag{8}$$

To prove this, we construct a specific sequence $x(m)$ related to $y(m)$. Let $y(m)$ be a solution of (2). For $m > m_0 + 1$, we define the input $u(m) = y(m+1)\|y(m+1)\|^{-1}\phi(m)$, where $\phi(m)$ is a scalar weighting function. By analyzing the relationship between the norms of $x(m)$ and $y(m)$, we derive the growth constraints on the evolution operator.

Specifically, if $u \in l_p$, we utilize the property that $\|x\| \leq K_1\|u\|_{l_p}$. Through a series of estimates involving the summation of the norms of $y(s)$, we obtain:

$$\|y(m_0+\tau+1)\|\Phi(m_0+\tau) < K\tau^{1/p} \tag{12}$$

where Φ is an auxiliary sum of the inverse norms of the trajectory. By applying logarithmic transformations and exponential bounds, we arrive at the final stability estimate (8). This result demonstrates that the "admissibility" of the pair (l_p, m)—meaning that every input in l_p produces a bounded output—implies a specific rate of decay for the homogeneous system.

2. Generalizations and Input Classes

The results can be extended to cases where the input u belongs to the space of bounded sequences m. If for every $u \in m$, the solution $x(m)$ is bounded, then the homogeneous system (2) is exponentially stable in the usual sense:

$$\|y(m)\| \leq N_1(m_0)\exp[-\alpha_1(m-m_0)]\|y(m_0)\| \tag{21}$$

where $\alpha_1 = \ln(1 + K^{-1})$. This confirms that the stability of the non-homogeneous system under bounded perturbations is equivalent to the exponential stability of the zero solution of the homogeneous system.

For a general linear system (1), the solution for $m > m_0$ can be written using the variation of constants formula:

$$x(m) = W(m, m_0)x(m_0) + \sum_{s=m_0+1}^{m} W(m, s)u(s-1) \tag{25}$$

If the operator $A(m)$ is uniformly bounded, $\|A(m)\| \leq A$, and the system satisfies the stability conditions derived above, then for any $u \in l_p$, the solution $x(m)$ satisfies:

$$\|x(m)\| \leq N_0 \exp[-\alpha_0(m - m_0)]\|x(m_0)\| + Q(m) \tag{34}$$

where $Q(m)$ represents the convolution of the evolution operator with the input sequence. We show that $Q(m)$ remains bounded for all $m > m_0$, ensuring the overall stability of the system.

3. Systems with Constant Coefficients

In the special case where $A(m) = A$ is a constant operator, the evolution operator simplifies to $W(m, s) = A^{m-s}$. The stability of the system is then determined by the spectral radius of A. If the spectral radius $\rho(A) < 1$, then $\|A^k\| \leq M\lambda^k$ for some $\lambda < 1$. The non-homogeneous system (1) with constant A is stable for any $u \in l_p$ if and only if the eigenvalues of A lie within the unit circle of the complex plane.

The estimates provided in this paper generalize classical results by Massera and Schäffer to the discrete-time case and provide explicit bounds for the decay rates $N(m_0)$ and α based on the properties of the input-output mapping.

References

1. Massera, J. L., & Schäffer, J. J. (1966). *Linear Differential Equations and Function Spaces.* Academic Press.
2. Daletskii, Yu. L., & Krein, M. G. (1970). *Stability of Solutions of Differential Equations in Banach Space.* Nauka.
3. Barbashin, E. A. (1967). *Introduction to the Theory of Stability.* Nauka.
4. Perron, O. (1930). Die Stabilitätsfrage bei Differentialgleichungen. *Mathematische Zeitschrift*, 32(1), 703-728.

Note: Figure translations are in progress. See original paper for figures.

Source: RussiaRxiv –Machine translation. Verify with original.
