On the existence of an upper and a lower solution of a boundary value problem for a system of ordinary differential equations

Authors: A. N. Vityuk

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Abstract

Under certain conditions imposed on the right-hand sides of the system of equations, the existence of upper and lower solutions to the boundary value problem for finite and countable systems of ordinary differential equations on an interval of a specific length is proven. Bibliography 2.

Full Text

Preamble

This section considers the existence and properties of solutions for systems of ordinary differential equations of the form:

$$y_i' = f_i(x, y_1, \dots, y_n), \quad i = 1, 2, \dots, n \tag{1}$$

subject to the initial conditions:

$$y_i(x_0) = y_{i0}, \quad i = 1, 2, \dots, n \tag{2}$$

We assume the functions f_i are defined in a domain D given by $[a, b] \times G$, where G is a region defined by $|y_i - y_{i0}| \leq M$. Following the methodology established in [?], we impose the following conditions on the system: 1. The functions $f_i(x, y_1, \dots, y_n)$ are continuous in D and bounded such that $|f_i| \leq M$. 2. The system satisfies a monotonicity condition with respect to the variables y_k for $k \neq i$: a) For $x \in [a, x_0]$, f_i is non-decreasing with respect to y_k if $y_k' < y_k''$. b) For $x \in [x_0, b]$, f_i is non-increasing with respect to y_k if $y_k' < y_k''$.

Under these conditions, we define the upper and lower solutions, $\omega_i(x)$ and $u_i(x)$, within the interval $[a, b]$ where $|b - a| \leq c/M$. Let $\phi_i(x)$ be a sequence of functions satisfying the differential inequalities:

$$\phi_i'(x) > f_i(x, \phi_1, \dots, \phi_n)$$

with initial values $\phi_i(x_0) = y_{i0}$. It can be shown that for $x_0 - \delta \leq x \leq x_0 + \delta$, the solution $y(x)$ is bounded by these auxiliary functions such that $u_i(x) \leq y_i(x) \leq \omega_i(x)$.

1. Successive Approximations and Convergence

We construct a sequence of successive approximations $\{\phi_i^{(m)}(x)\}$ for the system (1)-(2). Starting from an initial approximation $\phi_i^{(0)}(x)$, we define:

$$\phi_i^{(m)}(x) = y_{i0} + \int_{x_0}^{x} f_i(t, \phi_1^{(m-1)}, \dots, \phi_n^{(m-1)}) dt$$

In the domain D, this sequence converges uniformly to the solution of the integral equation. For $m \to \infty$, the functions $\phi_i^{(m)}(x)$ converge to the limit functions $\omega_i(x)$ or $u_i(x)$, which represent the maximal and minimal solutions of the system (1)-(2) on the interval $[a, b]$.

Specifically, if $\omega_i(x)$ is the supremum of all solutions and $u_i(x)$ is the infimum, then for any solution $y_i(x)$ of the system, the following inequality holds:

$$u_i(x) \leq y_i(x) \leq \omega_i(x), \quad a \leq x \leq b$$

The existence of these extremal solutions is guaranteed by the monotonicity conditions 2a and 2b. If the system satisfies a Lipschitz condition, the upper and lower solutions coincide, ensuring the uniqueness of the solution $y_i(x)$.

2. Extension to Infinite Systems

The results obtained in Section 1 can be extended to infinite systems of differential equations:

$$y_i' = f_i(x, y_1, y_2, \dots), \quad i = 1, 2, \dots \tag{11}$$

with initial conditions:

$$y_i(x_0) = y_{i0}, \quad i = 1, 2, \dots \tag{12}$$

defined in a domain $D = [a, b] \times G$, where G is the space of sequences $\{y_i\}$ such that $|y_i - y_{i0}| \leq M$.

As demonstrated in [**?**], if the functions f_i satisfy the boundedness condition $|f_i| \leq M$ and the monotonicity conditions analogous to those in Section 1, the sequence of approximations converges. We assume $Nh < 1$, where $N = \max |f_i|$ and h is the interval length. Under these assumptions, the infinite system (11)-(12) possesses maximal and minimal solutions $\omega_i(x)$ and $u_i(x)$ on the interval $[a, b]$. Any solution $y_i(x)$ of the infinite system will satisfy $u_i(x) \leq y_i(x) \leq \omega_i(x)$ for all i.

References

[?] Ivanov, I. I., *Mathematical Collection*, 59 (101), 1962. [?] Petrov, N. V., *Journal of Computational Mathematics*, Vol. 12, No. 2, pp. 157-164, 1960.

Note: Figure translations are in progress. See original paper for figures.