ON THE INDEX AND NORMAL SOLVABILITY OF A GENERAL LINEAR BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF EQUATIONS OF COMPOSITE TYPE

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 517.944

MATHEMATICS

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ON THE INDEX AND NORMAL SOLVABILITY OF A GENERAL LINEAR BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF EQUATIONS OF COMPOSITE TYPE

(Presented by Academician I. N. Vekua on May 16, 1966)

Consider in the plane $z = x + iy$ a simply connected domain G, bounded by a simple closed Lyapunov curve γ having no segments parallel to the axis $x = 0$, and suppose that every straight line $x = \text{const}$ passing through the domain G intersects its boundary γ in exactly two points. Let M_1 and M_0 be the points of tangency of the curve γ with the straight lines $x = x_0$ and $x = x^0$ $(x_0 < x^0)$. Taking as positive on γ the direction of motion counterclockwise, denote by γ_1 the positively oriented arc M_0M_1, and by γ_2 the arc $\gamma - \gamma_1$. In the domain G consider the system of equations

$$\begin{gathered}\partial u/\partial y = A_0(z)u + \operatorname{Re}(B_0(z)v),\\ \partial v/\partial \bar{z} - q(z)\partial v/\partial z = A_1(z)u + B_1(z)v + C_1(z)\bar{v},\end{gathered} \tag{1}$$

in which $u(z)$ is a real-valued, $v(z)$ a complex-valued unknown function; $A_0(z)$ is real-valued; $q(z), B_0(z), A_1(z), B_1(z), C_1(z)$ are complex given functions in G, moreover the function $q(z) \in C_\nu^n(\bar{G})$ and satisfies the inequality $|q(z)| \le \text{const} < 1,\ z \in G$, while the remaining coefficients of system (1) belong to the class $C_\nu^{n-1}(\bar{G})$, $0 < \nu < 1$.

An arbitrary system of three first-order equations of the form

$$\frac{\partial u_i}{\partial x} - \sum_{j=1}^{3} \left\{ A_{ij}(x,y) \frac{\partial u_j}{\partial y} + B_{ij}(x,y) u_j \right\} = 0,$$

$i = 1, 2, 3$, with real coefficients, can be reduced to a system of the form (1) if the polynomial $P(\lambda) = \det(\lambda E - \|A_{ij}(x,y)\|)$ has at all points of the domain one real and two imaginary roots, i.e. when throughout the domain the system is of composite type.

The purpose of the present note is to study the following problem:

Problem $A_{m,n}$. Find solutions $u(z), v(z)$ of system (1) such that $u(z) \in C_\nu^m(\bar{G})$, $v(z) \in C_\nu^n(\bar{G})$, satisfying the boundary conditions:

$$L_{m,n}(u,v)\big|_\gamma = h(t), \qquad \widetilde{L}_{m,n}(u,v)\big|_{\gamma_1} = \tilde{h}(t), \tag{2}$$

where

$$L_{m,n}(u,v) \equiv \sum_{j=0}^{m} b_j(z) \frac{\partial^j u}{\partial x^j} + \operatorname{Re}\left(\sum_{j=0}^{n} a_j(z) \frac{\partial^j v}{\partial z^j} \right),$$

$$\widetilde{L}_{m,n}(u,v) \equiv \sum_{j=0}^{m} \tilde{b}_j(z) \frac{\partial^j u}{\partial x^j} + \operatorname{Re}\left(\sum_{j=0}^{n} \tilde{a}_j(z) \frac{\partial^j v}{\partial z^j} \right),$$

$b_j(t), \tilde{b}_j(t), h(t), \tilde{h}(t)$ are real-valued, $a_j(t), \tilde{a}_j(t)$ are complex functions, Hölder continuous.

Problem $A_{m,n}$ is the most general linear boundary-value problem for system (1), since, if system (1) is used, then the case in which the operators $L_{m,n}(u,v)$, $\widetilde{L}_{m,n}(u,v)$ have the most general form is reduced to it:

$$\sum_{0 \le i+j \le m} b_{ij}(z) \frac{\partial^{i+j} u}{\partial x^i \partial y^j} + \operatorname{Re}\left(\sum_{0 \le i+j \le n} a_{ij}(z) \frac{\partial^{i+j} v}{\partial z^i \partial \bar{z}^j} \right).$$

Let $\bar{\gamma}_j$ be the closure of the arc γ_j $(j = 1, 2)$. Then the following holds.

Theorem. *If the conditions are satisfied:*

$$\Delta(t) \neq 0 \quad \text{on the arc } \bar{\gamma}_1, \tag{$N_{m,n}$}$$

$$a_n(t) \neq 0 \quad \text{on the arc } \bar{\gamma}_2,$$

where

$$\Delta(t) = a_n(t)\tilde{b}_m(t) - b_m(t)\tilde{a}_n(t),$$

and at the points M_0, M_1 the equalities hold:

$$b_m = \tilde{a}_n = 0, \qquad \tilde{b}_m = 1, \tag{$D_{m,n}$}$$

then for the solvability of the problem $A_{m,n}$ it is necessary and sufficient that a finite number of conditions be fulfilled, and the corresponding homogeneous *problem* $\overset{0}{A}_{m,n}$ $(h(t) = \tilde{h}(t) \equiv 0)$ *has a finite number of linearly independent solutions; moreover, the index of the problem* $A_{m,n}$ *is computed by the formula*

$$\operatorname{Ind}(A_{m,n}) = 2(\chi + n) + m + 1,$$

where

$$\chi = \frac{1}{2\pi}\{\arg a^*(t)\}_\gamma, \qquad a^*(t) = \begin{cases} \Delta(t), & t \in \gamma_1, \\ a_n(t), & t \in \gamma_2. \end{cases}$$

The proof of this theorem rests on the study of one singular integro-functional equation. Here the presence of lower-order terms in the system (1) and in the operators $L_{m,n}$, $\tilde{L}_{m,n}$ produces a completely continuous perturbation in the corresponding integro-functional equation, which will not cause any change in the final result. Therefore, without loss of generality, in the proof we shall consider only the principal part of the system and the principal parts $L^0_{m,n}$, $\tilde{L}^0_{m,n}$ of the boundary operators $L_{m,n}$ and $\tilde{L}_{m,n}$, i.e., we shall assume that $A_0(z) \equiv B_0(z) \equiv A_1(z) \equiv B_1(z) \equiv C_1(z) \equiv 0$, $b_j(z) \equiv \tilde{b}_j(z) \equiv 0$ for $j \neq m$, and $a_j(z) \equiv \tilde{a}_j(z) \equiv 0$ for $j \neq n$. In this case $u = u^0(x)$, where $u^0(x)$ is an arbitrary real-valued function of class $C^m_v(x_0, x^0)$ of one variable x, while $v(z)$ is a general solution of class $C^n_v(\bar{G})$ of the Beltrami equation $\partial v/\partial\bar{z} - q(z)\partial v/\partial z = 0$. Condition $(N_{m,n})$ makes it possible, under our assumptions, to write the boundary conditions in the form

$$\left.\begin{aligned} \operatorname{Re}\{\Delta(t_0)v^{(n)}(t_0)\} &= h_0(t_0) \\ \frac{d^m u^0}{dx^m} + \operatorname{Re}\{d(t_0)v^{(n)}(t_0)\} &= h_1(t_0) \end{aligned}\right\} \quad t_0 \in \gamma_1; \tag{3}$$

$$b_m(t_0)\frac{d^m u^0}{dx^m} + \operatorname{Re}\{a_n(t_0)v^{(n)}(t_0)\} = h(t_0), \qquad t_0 \in \gamma_2,$$

$$(v^{(n)}(z) \equiv \partial^n v/\partial z^n),$$

where $d(t_0)$, $h_0(t_0)$, $h_1(t_0)$ are fully determined functions expressed in terms of the coefficients of the principal parts and the right-hand sides of the conditions (2); moreover, by virtue of the equality $(D_{m,n})$, at the points M_0 and M_1 we have: $\Delta = a_n$, $b_m = d = 0$. Let $t = t(s)$ be the equation of the contour γ, and let $\alpha(s)$ be the homeomorphism of the part γ_i onto γ_j $(i \neq j)$ of the contour γ, defined by the equation $x(\alpha(s)) = x(s)$. This homeomorphism is generated when the arcs γ_1, γ_2 are intersected by the family of straight lines $x = \text{const}$. Then from the second condition (3) we find

$$\frac{d^m u^0}{dx^m} = h_1[t(\alpha(s_0))] - \operatorname{Re}\{d[t(\alpha(s_0))]v^{(n)}[t(\alpha(s_0))]\}$$

for $s_0 \in \gamma_2$. Hence, substituting into the last condition (3), we eliminate $d^m u^0/dx^m$, and, combining the resulting relation with the first condition, to determine the function $v(z)$ we obtain the boundary condition on the contour γ

$$\operatorname{Re}\{a^*(s_0)v^{(n)}[t(s_0)] + b^*(s_0)v^{(n)}[t(\alpha(s_0))]\} = h^*(s_0), \tag{4}$$

where

$$a^*(s) = \begin{cases} \Delta[t(s)], & s \in \gamma_1, \\ a_n[t(s)], & s \in \gamma_2; \end{cases} \qquad b^*(s) = \begin{cases} 0, & s \in \gamma_1, \\ -b_m[t(s)]\, d[t(\alpha(s))], & s \in \gamma_2; \end{cases}$$

$$h^*(s) = \begin{cases} h_0[t(s)], & s \in \gamma_1, \\ -b_m[t(s)]\, h_1[t(\alpha(s))], & s \in \gamma_2. \end{cases}$$

Let $\zeta = \zeta(z)$ be a solution of class $C_\nu^n(\overline{G})$ of the Beltrami equation $\zeta_{\bar z} - q(z)\zeta_z = 0$, realizing a homeomorphic mapping of the domain G onto some domain G^* of the ζ-plane. Then, as is known (^{1}), $v(z) = \varphi[\zeta(z)]$, where $\varphi(\zeta)$ is a holomorphic function in the domain G^*, belonging to the class C_ν^n in the closure of G^*.

Using now the integral representation of I. N. Vekua (see (^{2}), p. 275)

$$v(z) = \varphi[\zeta(z)] = \int_\gamma \mu(s)\left(1 - \frac{\zeta(z)}{\zeta(t)}\right)^{n-1} \ln\left(1 - \frac{\zeta(z)}{\zeta(t)}\right) ds + \int_\gamma \mu(s)\, ds + iC,$$

we reduce the boundary-value problem (4) to the following singular integro-functional equation:

$$\begin{aligned} K(\mu) \equiv\ & \operatorname{Re} A^*(s_0)\mu(s_0) + \operatorname{Re} B^*(s_0)\mu(\alpha(s_0)) + \frac{1}{\pi} \operatorname{Im} A^*(s_0) \int_\gamma \frac{\mu(s)\, ds}{s - s_0} \\ & + \frac{1}{\pi} \operatorname{Im} B^*(s_0) \int_\gamma \frac{\mu(s)\, ds}{s - \alpha(s_0)} + T(\mu) = h^*(s_0) + Ch^{**}(s_0), \end{aligned} \tag{5}$$

where $T(\mu)$ is a completely continuous operator;

$$A^*(s) = \frac{\pi i(-1)^n (n-1)!\, a^*(s)}{\zeta^{\,n-1}(t)\, [t'(s) + q(t)\overline{t'(s)}]} \left(\frac{\partial \zeta}{\partial t}\right)^{n-1},$$

$$B^*(s) = \frac{\pi i(-1)^n (n-1)!\, b^*(s)}{\zeta^{\,n-1}[t(\alpha(s))]\, [t'(\alpha(s)) + q(t(\alpha(s)))\overline{t'(\alpha(s))}]} \left(\frac{\partial \zeta}{\partial t}\right)^{n-1}_{s=\alpha(s)}.$$

In view of the fact that at the points M_0 and M_1 $\Delta = a_n,\ b_m = d = 0$, it is easy to see that the coefficients of equation (5) are continuous at all points of the contour γ.

Since, by condition $(N_{m,n})$, $A^*(s) \neq 0$ for $s \in \gamma$, while $B^*(s) = 0$ for $s \in \gamma_1$, $B^*(\alpha(s)) = 0$ for $s \in \gamma_2$ by its construction, then, using the known formula for permuting singular integrals (the Poincaré-Bertrand formula) (see (^{2}), p. 124), it is easy to verify that the operator

$$\begin{aligned} R(\mu) \equiv \operatorname{Re} \frac{1}{A^*(s_0)}\, \mu(s_0) - \operatorname{Re} \frac{\overline{B^*(s_0)}}{A^*(s_0)A^*(\alpha(s_0))}\, \mu(\alpha(s_0)) \\ + \frac{1}{\pi} \operatorname{Im} \frac{1}{A^*(s_0)} \int_\gamma \frac{\mu(s)\, ds}{s - s_0} + \frac{1}{\pi} \operatorname{Im} \frac{\overline{B^*(s_0)}}{A^*(s_0)A^*(\alpha(s_0))} \int_\gamma \frac{\mu(s)\, ds}{s - \alpha(s_0)} \end{aligned}$$

is a regularizer for the operator K. Hence the normal solvability of the problem $A_{m,n}$ follows immediately.

In order to compute the index of the problem $A_{m,n}$, it suffices for us to compute the index of the characteristic functional operator

$$\begin{aligned} K^0(\mu) \equiv \operatorname{Re} A^*(s_0)\mu(s_0) + \operatorname{Re} B^*(s_0)\mu(\alpha(s_0)) \\ + \frac{1}{\pi} \operatorname{Im} A^*(s_0) \int_\gamma \frac{\mu(s)\, ds}{s - s_0} + \frac{1}{\pi} \operatorname{Im} B^*(s_0) \int_\gamma \frac{\mu(s)\, ds}{s - \alpha(s_0)}. \end{aligned}$$

But since $B^*(s) = 0,\ s \in \gamma_1,\ B^*(\alpha(s)) = 0,\ s \in \gamma_2$, the index of this operator depends only on the coefficient $A^*(s)$ and is equal to * (see (3))

$$\frac{1}{\pi}\{\arg \overline{A^*(s)}\}_\gamma = \frac{1}{\pi}\{\arg \overline{a^*(s)}\}_\gamma - \frac{1}{\pi} \left\{ \arg \overline{\xi^{\,n-1}(t)}\, t'(s) \left(1 + \overline{q(t)}\, \frac{t'(s)}{\overline{t'(s)}} \right) \right\}_\gamma =$$

$$= \frac{1}{\pi}\{\arg \overline{a^*(s)}\}_\gamma + 2n.$$

The index of problem (4) will be equal to

$$\frac{1}{\pi}\{\arg \overline{a^*(s)}\}_\gamma + 2n + 1,$$

and since the function $u^0(x)$ is determined through $v^{(n)}[t(s)]$ on the arc γ_1 (or γ_2) up to m arbitrary real constants $c_0, c_1 \dots, c_{m-1}$,

$$u^0(x(s)) = c_0 + c_1(s - s^0) + \cdots + c_{m-1}(s - s^0)^{m-1} + \int_{s^0}^{s} \frac{(s - s')^{m-1}}{(m-1)!} \frac{d^m u^0}{ds'^m} \, ds',$$

it is easy to see that the index of the problem A_m^n is equal to

$$\frac{1}{\pi}\{\arg \overline{a^*(s)}\}_\gamma + 2n + 1 + m,$$

as was required to prove.

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Received
29 IV 1966

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* We may assume that $0 \in G$ and $\xi(0) = 0$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.