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Abstract

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MATHEMATICS

T. A. Azlarov, Kh. Mansurov

On a One-Parameter Apparatus for the Approximation of Functions

(Presented by Academician A. N. Kolmogorov, 18 IV 1966)

1. Let C be the set of continuous functions defined on the interval $[0,1]$. In the present paper, for the approximation of $f(x) \in C$, the following one-parameter apparatus is considered and studied:

$$P_{n,\alpha}(f;x) = \frac{1}{\sigma_\alpha n^\beta} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left[1 - \left|\frac{k}{n} - x\right|^\alpha\right]^{n\gamma}, \tag{1}$$

where $\alpha \geqslant 1,\ \beta = \dfrac{\alpha}{\alpha+2},\ \gamma = \dfrac{2\alpha}{\alpha+2},\ \sigma_\alpha = \dfrac{2}{\alpha}\Gamma\left(\dfrac{1}{\alpha}\right)$. For $\alpha = 2$ this apparatus was introduced in work ([1]).

2. It is not difficult to prove that if $f \in C$, then

$$\lim_{n\to\infty} \sup_x |P_{n,\alpha}(f;x) - f(x)| = 0. \tag{2}$$

The order of approximation in (2) is established by the following theorem:

Theorem 1. If $f \in C$ and $\omega_f(\delta)$ is its modulus of continuity, then

$$|P_{n,\alpha}(f;x) - f(x)| \leqslant K_1(\alpha)\omega_f\left(\frac{1}{n^{1-\beta}}\right) + \frac{K_2}{n^{2\beta}} + \frac{K_3}{n^\beta}\omega_f\left(\frac{1}{n^{1-\beta}}\right), \tag{3}$$

where

$$K_1(\alpha) = 1 + \frac{2}{\sigma_\alpha}\int_1^\infty y e^{-y^\alpha}\, dy, \tag{4}$$

K_2, K_3 are constants independent of n.

The obtained estimate of approximation (3) cannot be improved in order (see Theorem 3).

For $\alpha = 2$, from (4) we obtain

$$K_1(2) = 1 + \frac{1}{e\sqrt{\pi}}.$$

(We note that the corresponding constant given in Theorem 1 of work ([2]) was equal to 1.5.)

The proof of Theorem 1 is based essentially on the following asymptotic equalities:

for $\alpha \neq 1$

$$P_{n,\alpha}(1;x) = 1 - \frac{\alpha+1}{2\alpha^2 n^{2\beta}} + O\left(n^{-\beta(1+\alpha)} + n^{-4\beta}\right); \tag{5}$$

for $\alpha = 1$

$$P_{n,1}(1,x) = 1 - \left[\frac{25}{24} - \frac{1}{2}\rho^2\left(nx + \frac{1}{2}\right)\right]\frac{1}{n^2} + O\left(n^{-4\beta}\right),$$

where $\rho(t)$ is the distance from the point t to the nearest integer.

Let $\omega_0(\delta)$ be some fixed modulus of continuity. Introduce the class C_{ω_0} of functions $f(x)$ for which

$$\omega_f(\delta) \leqslant \omega_0(\delta), \qquad 0 \leqslant \delta \leqslant 1.$$

Theorem 2. Put

$$E_{n,\alpha}(x, C_{\omega_0}) = \sup_{f \in C_{\omega_0}} \left|P_{n,\alpha}(f;x) - f(x)\right|.$$

Then, as $n \to \infty$,

$$E_{n,\alpha}(x, C_{\omega_0}) = \frac{1}{\sigma_\alpha n^\beta}\sum_{k=0}^{n}\omega_0\left(\left|\frac{k}{n} - x\right|\right)\left[1 - \left|\frac{k}{n} - x\right|^\alpha n^\gamma\right] + O(n^{-2\beta}). \tag{6}$$

Theorem 3. The following asymptotic formula holds ($n \to \infty$):

$$E_{n,\alpha}(xC_{\omega_0}) = \frac{2}{\sigma_\alpha}\int_0^\infty \tilde{\omega}\left(\frac{y}{n^{1-\beta}}\right)e^{-y^\alpha}\,dy + O\left(\frac{1}{n^{2\beta}}\right) + O\left(\omega_0\left(\frac{1}{n}\right)\right). \tag{7}$$

Here $\tilde{\omega}(\delta) = \omega_0(\delta)$ if $0 \leqslant \delta \leqslant 1$, and $\tilde{\omega}(\delta) = 0$ for $\delta > 1$.

Proof of Theorem 3. Denoting

$$\widetilde{E}_{n,\alpha}(x, C_{\omega_0}) = \frac{1}{\sigma_\alpha n^\beta} \sum_{k=0}^{n} \omega_0 \left(\left| \frac{k}{n} - x \right| \right) \left[1 - \left| \frac{k}{n} - x \right|^\alpha n^\gamma \right],$$

from (6) we obtain

$$E_{n,\alpha}(x, C_{\omega_0}) = \widetilde{E}_{n,\alpha}(x, C_{\omega_0}) + O(n^{-2\beta}).$$

Next, put $\delta_n = n^{-\beta/\alpha}$ and $\widetilde{E}_{n,\alpha}(x, C_{\omega_0}) = S_1 + S_2$, where in S_1 the summation is over all k for which $|k/n - x| < \delta_n$, and in S_2 over the remaining k. It is not hard to see that

$$S_2 = O(n^{1-\beta} e^{-n^\beta}).$$

Now consider

$$S_1 = \frac{1}{\sigma_\alpha} \sum_{|k/n - x| < \delta_n} \omega_0 \left(\frac{|\xi_k|}{n^{1-\beta}} \right) \left[1 - \frac{|\xi_k|^\alpha}{n^\gamma} \right]^{n^\gamma} \Delta\xi_k,$$

where

$$\xi_k = (k - nx)/n^\beta, \qquad \Delta\xi_k = \xi_{k+1} - \xi_k.$$

Since, as $n \to \infty$, $|\xi|^\alpha n^{-\gamma} \leqslant \delta_n^\alpha = n^{-\beta} \to 0$, we may use the expansion

$$\left(1 - \frac{|\xi_k|^\alpha}{n^\gamma} \right)^{n^\gamma} = e^{-|\xi_k|^\alpha} \left[1 + O\left(|\xi_k|^{2\alpha} n^{-\gamma} \right) \right].$$

We have

$$S_1 = \frac{1}{\sigma_\alpha} \sum_{|\xi_k| < \delta_n n^{1-\beta}} \omega_0 \left(\frac{|\xi_k|}{n^{1-\beta}} \right) e^{-|\xi_k|^\alpha} \Delta\xi_k + O\left(n^{-\gamma} \sum_{k=0}^{n} \omega_0 \left(\frac{|\xi_k|}{n^{1-\beta}} \right) e^{-|\xi_k|^\alpha} |\xi_k|^{2\alpha} \Delta\xi_k \right). \tag{8}$$

The second term on the right-hand side of the last equality is a quantity of order $O(n^{-2\beta})$.

Let

$$R_k = \omega_0\left(\frac{|\xi_k|}{n^{1-\beta}}\right)e^{-|\xi_k|^\alpha}\Delta\xi_k - \int_{\xi_k-\frac{1}{2}n^{-\beta}}^{\xi_k+\frac{1}{2}n^{-\beta}}\omega_0\left(\frac{|t|}{n^{1-\beta}}\right)e^{-|t|^\alpha}\,dt, \tag{9}$$

$$R = \sum_{|\xi_k|<n^{\beta/\alpha}} R_k. \tag{10}$$

Using the known properties of the modulus of continuity: $\omega(\lambda\delta) \le (1+\lambda)\omega(\delta)$ and $\omega(t_1) - \omega(t_2) \le \omega(|t_1 - t_2|)$, one can show that

$$|R_k| \le \omega_0\left(\frac{1}{n}\right)e^{-|\xi_k|^\alpha}Q(|\xi_k|)\Delta\xi_k, \tag{11}$$

where $Q(z)$ is some polynomial in z.

From relations (8), (9), (10), and (11) it follows that

$$S_1 = \frac{1}{\sigma_\alpha}\int_{-n^{\beta/\alpha}}^{n^{\beta/\alpha}}\omega_0\left(\frac{|y|}{n^{1-\beta}}\right)e^{-|y|^\alpha}\,dy + O\left(\omega_0\left(\frac{1}{n}\right)\right) + O(n^{-2\beta}).$$

Hence, taking into account the estimate for S_2, it is easy to obtain the assertion of the theorem.

Remark. 1) As is known, the problem of the asymptotics of the upper bound for the deviation of functions from their approximating apparatus for various classes of functions was considered by A. N. Kolmogorov, S. M. Nikol' skii, and others. The formulated theorem solves this problem for the apparatus (1).

2) It is interesting to note that the principal term of expansion (7) does not depend on x.

3) If, for example, for small x we have the expansion

$$\omega_0(x) = x^\lambda(a_0 + a_1x + \cdots), \qquad 0 < \lambda \le 1,$$

then for any integer s ($s \ge 0$)

$$\frac{2}{\sigma_\alpha}\int_0^\infty \tilde{\omega}\left(\frac{y}{n^{1-\beta}}\right)e^{-y^\alpha}\,dy = \frac{2}{\sigma_\alpha}n^{1-\beta}\int_0^1 \omega_0(x)e^{-n^{2\beta}x^\alpha}\,dx =$$

$$= \frac{1}{\Gamma(1/\alpha)}\sum_{k=0}^{s} a_k\Gamma\left(\frac{\lambda+k+1}{\alpha}\right)n^{-2\frac{\lambda+k}{\alpha+2}} + o\left(n^{-2\frac{\lambda+s}{\alpha+2}}\right).$$

Hence it is not difficult to obtain that:

for $\lambda \geq \dfrac{2\alpha}{\alpha+2}$,

$$E_{n,\alpha}(x, C_{\omega_0}) = \frac{1}{\Gamma(1/\alpha)} a_0 \Gamma\left(\frac{\lambda+1}{\alpha}\right) n^{-2\frac{\lambda}{\alpha+2}} + O\left(n^{-2\frac{\alpha}{\alpha+2}}\right);$$

for $\lambda < \dfrac{2\alpha}{\alpha+2}$,

$$E_{n,\alpha}(x, C_{\omega_0}) = \frac{1}{\Gamma(1/\alpha)} \sum_{k=0}^{s} a_k \Gamma\left(\frac{\lambda+k+1}{\alpha}\right) n^{-\frac{2(\lambda+k)}{\alpha+2}} + O(n^{-\lambda}),$$

where $s = [\lambda\alpha/2]$.

3. Questions on the convergence of $P_{n,2}(f;x)$ to $f(x)$ and on the estimate of this approximation in terms of the modulus of continuity ω_f were considered in ([1,2]). In ([2]), moreover, for $f \in C^2$ (C^m is the class of functions having an m-th continuous derivative), the asymptotic formula

$$P_{n,2}(f;x) = f(x) + \frac{f''(x) - 1.5f(x)}{4n} + o\left(\frac{1}{n}\right). \qquad (*)$$

was proved. If we consider

$$\overline{P}_{n,2}(f;x) = \frac{8n}{8n-3} P_{n,2}(f;x),$$

then formula (*) takes the form

$$\overline{P}_{n,2}(f;x) = f(x) + \frac{f''(x)}{4n} + o\left(\frac{1}{n}\right).$$

If one carries out an analogous "shift of normalization" in the general case and considers the operator

$$\overline{P}_{n,\alpha}(f;x) = \frac{P_{n,\alpha}(f;x)}{1 - \dfrac{\alpha+1}{2\alpha^2} n^{-2\beta}},$$

then one can prove the theorem:

Theorem 4. Let $f \in C^2$ and $\alpha > (\sqrt{17}-1)/2$. Then

$$\overline{P}_{n,\alpha}(f;x) = f(x) + \frac{\Gamma(3/\alpha)}{2\Gamma(1/\alpha)} \frac{f''(x)}{n^{2-2\beta}} + o\left(\frac{1}{n^{2-2\beta}}\right).$$

4. For $\alpha = 2l$, $l = 1, 2, \dots$, it is not difficult to prove that

$$\lim_{n\to\infty} \sup_x \left| \frac{\partial P_{n,\alpha}(f;x)}{\partial x} - \frac{df}{dx} \right| = 0$$

and to estimate the error of this approximation. Under the assumption that $f \in C^1$ and ω_{1f} is the modulus of continuity of $f'(x)$, Theorems 1–3 can be formulated in terms of ω_{1f}.

In conclusion, we note that the results obtained also extend to functions depending on many variables.

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Tashkent State University
named after V. I. Lenin

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Note: Figure translations are in progress. See original paper for figures.