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In the spring of 1966 the late N. V. Smirnov posed to us the following combinatorial problem, important for the theory of multidimensional order statistics.

Suppose there are n objects belonging to $s+1$ different types (states) A_i, and that r_i objects belong to type A_i,

$$\sum_{i=1}^{s+1} r_i = n.$$

It is required to determine the number of ways in which these objects can be arranged in a chain so that each object of type A_i is followed by an object of type A_k, $k \neq i$.

1. In [1] the number was found of chains of length n in which all numbers of transitions m_{ij} from state A_i to state A_j, beginning with A_{i_0} and ending with state A_{j_0}, are fixed. If the external transition from the last state of the chain to the first is included in the number $m_{j_0 i_0}$, then this number (see formula (6) in [1]) takes the form

$$K_{i_0 j_0}^{(s+1)}(r_i, m_{ij}) = \frac{m_{j_0 i_0}}{\prod_{i=1}^{s+1} r_i} \frac{\prod_{i=1}^{s+1} r_i!}{\prod_{i,j=1}^{s+1} m_{ij}!} \begin{vmatrix} r_1 - m_{11} & -m_{12} & \dots & -m_{1s} \\ -m_{21} & r_2 - m_{22} & \dots & -m_{2s} \\ \dots & \dots & \dots & \dots \\ -m_{s1} & -m_{s2} & \dots & r_s - m_{ss} \end{vmatrix}, \tag{1}$$

where

$$\sum_{i=1}^{s+1} m_{ij} = r_j, \qquad \sum_{j=1}^{s+1} m_{ij} = r_i, \qquad i, j = 1, 2, \dots, s+1. \tag{2}$$

In order to find the number of chains $M_1^{(s+1)}(r_i, m_{ij})$ in which no two identical states stand next to each other and the ends of the chains belong to different states, it suffices to put all $m_{ii} = 0$ in (1) and sum over all i_0, j_0, $i_0 \neq j_0$; thus,

$$M_1^{(s+1)}(r_i, m_{ij}) = \frac{n \prod_{i=1}^{s+1} r_i!}{\prod_{i=1}^{s+1} r_i \prod_{\substack{i,j=1\\ i\neq j}}^{s+1} m_{ij}!} \begin{vmatrix} r_1 & -m_{12} & \dots & -m_{1s} \\ -m_{21} & r_2 & \dots & -m_{2s} \\ \dots & \dots & \dots & \dots \\ -m_{s1} & -m_{s2} & \dots & r_s \end{vmatrix}. \qquad (3)$$

To obtain the analogous number of chains $M_2^{(s+1)}(r_i, m_{ij})$ with identical ends, one must put in (1) $j_0 = i_0$, $m_{i_0 i_0} = 1$, $m_{ii} = 0$ for $i \neq i_0$, and sum over i_0 from 1 to $s+1$.

Thus, the formulated problem has the following exact solution:

$$M^{(s+1)}(r_1, r_2, \dots, r_{s+1}) = \sum_{\substack{m_{ij}\\ i\neq j}} \left[M_1^{(s+1)}(r_i, m_{ij}) + M_2^{(s+1)}(r_i, m_{ij}) \right]. \qquad (4)$$

Remark. If we now sum the expressions found over all r_i with fixed sum

$$\sum_{i=1}^{s+1} r_i = n,$$

then we obtain the following pro-

the formulas ($s \geqslant 1$):

$$M_1^{(s+1)}(n) = \sum_{m_{ij}, r_i} M_1^{(s+1)}(r_i, m_{ij}) = s^n + (-1)^n s,$$

$$M_2^{(s+1)}(n) = \sum_{m_{ij}, r_i} M_2^{(s+1)}(r_i, m_{ij}) = s^{n-1} + (-1)^{n-1} s, \qquad (5)$$

which follow from the recurrence relations

$$M_1^{(s+1)}(n+1) = s M_2^{(s+1)}(n) + (s-1) M_1^{(s+1)}(n),$$

$$M_2^{(s+1)}(n+1) = M_1^{(s+1)}(n). \qquad (6)$$

From (6), in particular, the obvious expression follows

$$M^{(s+1)}(n) = M_1^{(s+1)}(n) + M_2^{(s+1)}(n) = (s+1) s^{n-1}. \qquad (7)$$

2. We shall dwell in more detail on the computation of the asymptotic value of the first part of the sum (4), corresponding to the number of all chains with different ends. We need to sum expression (3) over m_{ij} under the conditions $m_{ii} = 0$ and conditions (2); in this case there are $s^2 - s - 1$ independent variables, or degrees of freedom.

We shall call a function $f(x) = f(x_1, x_2, \dots, x_k)$ slowly increasing if, as $x_i \to \infty$, $f(x + \varphi(x))/f(x) \to 1$, where $\varphi_i(x_i) \to \infty$, and $\varphi_i(x_i)/x_i \to 0$, $i = 1, 2, \dots, k$.

We need to find the asymptotic value of a sum whose terms, according to (3), have the form $f(x_1, x_2, \dots, x_k) \prod_{i=1}^{k} C_{a_i}^{x_i}$, where $f(x_1, x_2, \dots, x_k)$ is a slowly increasing function of $s^2 - s - 1$ independent variables, and the a_i are linear functions of $x_1, x_2, \dots, x_k$.

The summation method is as follows. For the product of binomial coefficients, a stationary point $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k)$ is sought, at which this product attains its greatest value. Setting $x_i = \bar{x}_i + t_i\sqrt{r}$, where $r = \min(r_1, r_2, \dots, r_{s+1})$, and applying Stirling’ s formula for sufficiently large r, we obtain

$$\sum_{x_i} f(x_1, x_2, \dots, x_k) \prod_{i=1}^{k} C_{a_i}^{x_i} = f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k) \prod_{i=1}^{k} C_{\bar{a}i}^{\bar{x}_i} (2\pi r)^{(s^2-s-1)/2} \frac{1 + o(1)}{\sqrt{\Delta}} \tag{8}$$

where Δ is the determinant of the positive definite quadratic form $A(t, t)$, which is obtained by separating out the principal part of the logarithms of the binomial coefficients in a neighborhood of the stationary point. In finding the stationary point, the slowly increasing function (in our case the role of this function is played by the determinant entering into (3)) may be disregarded.

In our case, for $s > 2$, the coordinates of the stationary point are found from a system of $s^2 - s - 1$ nonlinear equations, and for arbitrary r_i their expression is rather cumbersome. We give here the asymptotic formulas in two cases: for $s = 2$ and arbitrary r_1, r_2, r_3, for which the problem posed has a nonzero solution, and for any s in the uniform case, when $r_1 = r_2 = \dots = r_{s+1} = n/(s+1)$.

3. Put $s = 2$ (the case of three states). The chains of interest to us can be composed only under the conditions

$$r_1 + r_2 - r_3 \geqslant 0, \qquad r_1 + r_3 - r_2 \geqslant 0, \qquad r_2 + r_3 - r_1 \geqslant 0.$$

The coordinates of the stationary point in this case are found from a linear equation and have the form

$$\bar{m}_{12} = \bar{m}_{21} = (r_1 + r_2 - r_3)/2, \quad \bar{m}_{13} = \bar{m}_{31} = (r_1 + r_3 - r_2)/2,$$
$$\bar{m}_{23} = \bar{m}_{32} = (r_2 + r_3 - r_1)/2. \tag{9}$$

If we introduce the notation

$$F(r_1, r_2, r_3) = \frac{\sqrt{2\pi r_1! r_2! r_3!}\sqrt{(n^2 - 2r_1^2 - 2r_2^2 - 2r_3^2)(n - 2r_1)(n - 2r_2)(n - 2r_3)}}{8 r_1 r_2 r_3 \Gamma^2(1 + n/2 - r_1)\Gamma^2(1 + n/2 - r_2)\Gamma^2(1 + n/2 - r_3)}, \tag{10}$$

then

$$M_1^{(3)}(r_1, r_2, r_3) = \sum_{m_{ij}} M_1^{(3)}(r_i, m_{ij}) = nF(r_1, r_2, r_3)(1 + o(1)), \tag{11}$$

$$M_2^{(3)}(r_1, r_2, r_3) = \sum_{m_{ij}} M_2^{(3)}(r_i, m_{ij}) = \tag{12}$$

$$= \{r_1 F(r_1 - 1, r_2, r_3) + r_2 F(r_1, r_2 - 1, r_3) + r_3 F(r_1, r_2, r_3 - 1)\}(1 + o(1))$$

and the number of all chains of the type that interests us, composed of objects of three kinds, is equal to

$$M^{(3)}(r_1, r_2, r_3) = M_1^{(3)}(r_1, r_2, r_3) + M_2^{(3)}(r_1, r_2, r_3). \tag{13}$$

4. Let s be arbitrary ($s \geq 2$) and $r_1 = r_2 = \ldots = r_{s+1} = n/(s+1) = r$. In this case the stationary point is found without difficulty,

$$\bar{m}_{ij} = n/s(s+1) = r/s, \quad i \neq j,$$

and the determinant of the quadratic form from (8) has the form

$$\Delta = s^{s^2-s}(s-1)(s^2-1)^{s-1}, \tag{14}$$

the number of degrees of freedom is $s^2 - s - 1$, and

$$M_1^{(s+1)}(r, r, \ldots, r) = \tag{15}$$

$$= \frac{(s+1)r(r!)^{s+1}}{r^{s+1}[\Gamma(1 + r/s)]^{s^2+s}} \begin{vmatrix} r & -r/s & \ldots & -r/s \\ -r/s & r & \ldots & -r/s \\ \ldots & \ldots & \ldots & \ldots \\ -r/s & -r/s & \ldots & r \end{vmatrix} \left\{ \frac{(2\pi r)^{s^2-s-1}}{s^{s^2-s}(s-1)(s^2-1)^{s-1}} \right\}^{1/2} (1+o(1)),$$

or, after elementary transformations,

$$M_1^{(s+1)}(r, r, \ldots, r) = \left(2\pi \frac{n}{s+1}\right)^{-s/2} \left(\frac{s+1}{s-1}\right)^{s/2} \sqrt{s+1}\, s^n (1 + o(1)). \tag{16}$$

It is easy to show that the number of chains with identical ends, as in formula (5), will be (asymptotically) s times smaller, i.e.

$$M_2^{(s+1)}(r, r, \ldots, r) = \frac{1}{s} M_1^{(s+1)}(r, r, \ldots, r)(1 + o(1)). \tag{17}$$

5. Let us note that the number of chains $M^{(s+1)}(r, r, \dots, r)$ of our kind is the number of chains containing exactly n runs, and finding it is important for the theory of distributions of runs in successive trials with $s+1$ outcomes.

In general, if it is necessary to determine the number $N^{(s+1)}(r_i, l_i)$ of chains with prescribed numbers of appearances of the states $r_1, r_2, \dots, r_{s+1}$ and prescribed numbers of runs $l_1, l_2, \dots, l_{s+1}$ of each of these states, then this problem immediately obtains the solution

$$N^{(s+1)}(r_1, r_2, \dots, r_{s+1}; l_1, l_2, \dots, l_{s+1}) = M^{(s+1)}(l_1, l_2, \dots, l_{s+1}) \sum_{i=1}^{s+1} C_{r_i-1}^{l_i-1}, \quad (18)$$

since from r_i states, l_i runs can be formed in a number of ways equal to

$$C_{r_i-1}^{l_i-1}.$$

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REFERENCES

1. N. V. Smirnov, O. V. Sarmanov, V. K. Zakharov, DAN, **167**, No. 6, 1238 (1966).

Note: Figure translations are in progress. See original paper for figures.