

A LIMIT THEOREM FOR THE NUMBER OF LEVEL CROSSINGS BY A STATIONARY GAUSSIAN PROCESS

MATHEMATICS

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.33857>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.21

MATHEMATICS

M. P. ERSHOV

A LIMIT THEOREM FOR THE NUMBER OF LEVEL CROSSINGS BY A STATIONARY GAUSSIAN PROCESS

(Presented by Academician Yu. V. Linnik on 28 II 1967)

Consider a continuous with probability 1 stationary Gaussian process ξ_t with correlation function r_t . Let ξ_t be differentiable in mean square (m.s.), and let $N(s, t)$ denote the number of crossings from below to above of the level u by the process ξ_t on $[s, t]$. From ⁽¹⁾ it follows that $EN(s, t) < \infty$.

Lemma 1. *If $r_t \rightarrow 0$ as $t \rightarrow \infty$, then*

$$DN(0, T) \xrightarrow{T \rightarrow \infty} \infty$$

for all u .

Proof. Let $DN(0, T) < \infty$, otherwise the assertion of the lemma is trivial. Put

$$X_k(\tau) = N(k\tau, (k+1)\tau) - EN(k\tau, (k+1)\tau), \quad k = 0, 1, \dots \quad (1)$$

For any τ , $\{X_k(\tau)\}$ is a stationary sequence, and, by Theorem 2 on p. 119 in ⁽²⁾, $EX_0 X_k \rightarrow 0$ as $k \rightarrow \infty$. Choose τ so that

$$EN(0, \tau) = 1/2. \quad (2)$$

Let $\varliminf_n DN(0, n\tau) < \infty$. Then, by Theorem 18.2.2 in ⁽³⁾, there is a representation:

$$X_k = Y_{k+1} - Y_k,$$

where $Y_k = U^k Y_0$, $Y_0 \in H(X)$, and $H(X)$ is the closure, in the m.s. sense, of the linear span of the set $\{X_0, X_1, \dots\}$, while U is a unitary operator in $H(X)$ defining X_k : $X_k = U^k X_0$. From (2) it follows that

$$\mathbf{P}\{|Y_{2n+1} - Y_0| < 1/2\} = 0. \quad (3)$$

Let $I = (a, a + 1/2)$ be an interval such that $\mathbf{P}\{Y_0 \in I\} = p \neq 0$. The variables Y_n , obviously, are measurable with respect to ξ_t , $0 \leq t < \infty$; therefore, by the mentioned Theorem 2 in (2),

$$\mathbf{P}\{Y_0, Y_n \in I\} \xrightarrow[n \rightarrow \infty]{} p^2,$$

which contradicts (3). Thus, $\lim_n DN(0, n\tau) = \infty$. Let $T \rightarrow \infty$, and put $n = [T/\tau]$. We have

$$DN(0, T) > DN(0, n\tau) - 2\sqrt{DN(0, n\tau) E(N(0, \tau))^2} \rightarrow \infty.$$

Lemma 2. *If ξ_t is twice differentiable in the m.s. sense and*

$$\int_0^\infty t(|r_t| + |r'_t| + |r''_t|) dt < \infty,$$

then, as $T \rightarrow \infty$,

$$DN(0, T) = \sigma^2 T(1 + o(1))$$

and $\sigma \neq 0$.

Proof. From (5) it follows that $DN(0, T) < \infty$. Let τ be fixed.

$$DN(0, n\tau) = nEX_0^2(\tau) + 2 \sum_{k=1}^{n-1} (n-k)EX_0(\tau)X_k(\tau), \quad (4)$$

where $X_k(\tau)$ is defined from (1). In the same way as in (4) when finding the factorial moments of $N(0, T)$, one can show that

$$\mathbf{E}X_0(\tau)X_k(\tau) = \int_0^\tau \left(\int_{k\tau}^{(k+1)\tau} (F_{t-s} - F_\infty) dt \right) ds, \quad k > 0, \quad (5)$$

where

$$F_{t-s} = \int_0^\infty \int_0^\infty y_1 y_2 \varphi_{t-s}(u, u, y_1, y_2) dy_1 dy_2, \quad F_\infty = \lim_{t \rightarrow \infty} F_t,$$

and $\varphi_{t-s}(x_1, x_2, y_1, y_2)$ is the density of the vector $\{\xi_s, \xi_t, \dot{\xi}_s, \dot{\xi}_t\}$. (We note that, by the conditions of the lemma, $r_t, r'_t, r''_t \rightarrow 0$ as $t \rightarrow \infty$, so the definition of F_∞ makes sense.) Let R_{t-s} and R_∞ be the correlation matrices corresponding to φ_{t-s} and $\varphi_\infty = \lim_{t \rightarrow \infty} \varphi_t$. Put $\delta_{t-s} = \|R_{t-s} - R_\infty\|$, where $\|(a_{ij})\| = \max_{ij} |a_{ij}|$. It is easy to obtain the estimate

$$|\varphi_{t-s}(u, u, y_1, y_2) - \varphi_\infty(u, u, y_1, y_2)| < C \exp \left\{ -(4r_0 - 4r''_0)^{-1} (y_1^2 + y_2^2) \delta_{t-s} |R_{t-s}|^{-3/2} (1 + y_1^2 + y_2^2) \right\}, \quad (6)$$

where C depends only on u, r_0, r''_0 . From (5) and (6) we obtain

$$\begin{aligned} \sum_1^\infty |\mathbf{E}X_0 X_k| &< 2 \int_0^\infty t |F_t - F_\infty| dt < \\ &< 2a \int_0^a |F_t - F_\infty| dt + 2C_1 \int_a^\infty t \delta_t |R_t|^{-3/2} dt. \end{aligned} \quad (7)$$

The first integral is finite, since $DN(0, a) < \infty$. It is obvious that $|R_t|$ is bounded below uniformly in t on $[a, \infty)$ for any $a > 0$. In addition, $\delta_t < |r_t| + |r'_t| + |r''_t|$; therefore the second integral on the right-hand side of (7) is also finite. Finally we obtain

$$\left| \sum_1^\infty \mathbf{E}X_0(\tau) X_k(\tau) \right| < K,$$

where K does not depend on τ . By Lemma 1, $\mathbf{E}X_0^2(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. Choose τ so that $\mathbf{E}X_0^2(\tau) > 2K$. Now (4) can be rewritten as follows:

$$\begin{aligned} DN(0, n\tau) &= n \left(\mathbf{E}X_0^2(\tau) + 2 \sum_1^\infty \mathbf{E}X_0(\tau) X_k(\tau) + o(1) \right) = \\ &= n\sigma_1^2(1 + o(1)), \quad \sigma_1 \neq 0. \end{aligned}$$

Put $\sigma^2 = \sigma_1^2/\tau$, and let $T \rightarrow \infty$, $n = [T/\tau]$.

$$\begin{aligned} \left| \frac{DN(0, T)}{\sigma^2 T} - 1 \right| &< \frac{|DN(0, T) - DN(0, n\tau)|}{\sigma^2 T} + \\ &+ DN(0, n\tau) \left(\frac{1}{\sigma^2 n\tau} - \frac{1}{\sigma^2 T} \right) + \left| \frac{DN(0, n\tau)}{\sigma^2 n\tau} - 1 \right| = o(1). \end{aligned}$$

Lemma 3. Let R_t denote the correlation matrix of the vector $\{\xi_0, \xi_t, \dot{\xi}_0, \dot{\xi}_t\}$. If $r_t^{(8)}$ is continuous, then in some neighborhood of zero

$$|R_t| = ct^8 + o(t^8), \quad c > 0.$$

Proof. Putting $r_0 = 1$, we find

$$|R_t| = (r_t^2 + r_0'' - r_t r_t'')^2 - (r_t' - r_t r_0'')^2.$$

It is enough to expand the terms in the parentheses up to t^6 , since the expansion will begin with t^2 . We obtain

$$|R_t| = \frac{t^8}{144} (r_0^{(4)} - r_0''^2) (r_0^{(6)} r_0'' - r_0^{(4)2}) + o(t^8).$$

From the positive definiteness of R_t it follows that the coefficient of t^8 is nonnegative. Let $r_0^{(4)} = r_0''/2$. This means that $E\xi_0^2 E\dot{\xi}_0^2 = (E\xi_0 \dot{\xi}_0)^2$. But in the range space of ξ_t , which is the closure in mean square of the linear span of the variables ξ_{t_k} , such an equality is possible only under the condition $E(\dot{\xi}_0 - \lambda \xi_0)^2 = 0$, where λ is some number; this cannot occur when the process ξ_t is Gaussian. Similarly we verify that $r_0^{(6)} r_0'' - r_0^{(4)2} > 0$.

Lemma 4. If there exists and is continuous in mean square $d^4 \xi_t / dt^4$, and $r_t, r_t', r_t'' = O(t^{-\alpha})$ as $t \rightarrow \infty$ for some $\alpha > 13$, then for all u

$$\mu_4(T) = E(N(0, T) - EN(0, T))^4 = O(T^2) \quad \text{as } T \rightarrow \infty.$$

Proof. From (5) it follows that $\mu_4(T) < \infty$. In the notation (1) we have

$$\begin{aligned} \mu_4(n\tau) &< nEX_0^2 + \sum_{i \neq j} |E(X_i^2 X_j^2 + X_i^3 X_j)| + \sum_{i \neq j \neq k} |EX_i^2 X_j X_k| \\ &+ \sum_{i \neq j \neq k \neq l} |EX_i X_j X_k X_l| \\ &= O\left(n^2 + \sum_{i < j-1 < k-2} |EX_i^2 X_j X_k| + \sum_{i < j-1 < k-2 < l-3} |EX_i X_j X_k X_l|\right). \end{aligned} \quad (8)$$

Put $N(i\tau, (i+1)\tau) = N_i$, $N_i(N_i - 1) = N_i^{(2)}$. Consider the second term on the right-hand side of (8):

$$EX_i^2 X_j X_k = E(N_i^{(2)} N_j + N_i N_j - (N_i^{(2)} + N_i) EN_j - (EN_i)^2 N_j) X_k. \quad (9)$$

As in (4), we obtain:

$$EN_i^{(2)} N_j X_k = \int_{\Delta_i \times \Delta_i} dt_1 dt_2 \int_{\Delta_j} dt_3 \int_{\Delta_k} (F_{t_1 t_2 t_3 t_4} - F_{t_1 t_2 t_3 \infty}) dt_4, \quad (10)$$

where

$$F_{t_1 t_2 t_3 t_4} = \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_4 \varphi_{t_1 \dots t_4}(u, \dots, u, y_1, \dots, y_4) dy_1 \cdots dy_4$$

and $\varphi_{t_1 \dots t_4}(x_1, \dots, x_4, y_1, \dots, y_4)$ is the density of the vector $\{\xi_{t_1}, \dots, \xi_{t_4}, \dot{\xi}_{t_1}, \dots, \dot{\xi}_{t_4}\}$, $F_{t_1 t_2 t_3 \infty} = \lim_{t_4 \rightarrow \infty} F_{t_1 t_2 t_3 t_4}$, with $t_1 \neq t_2 \in \Delta_i = (i\tau, (i+1)\tau)$, $t_3 \in \Delta_j$. Denote by $R_{t_2-t_1}$ the correlation matrix of the vector $\{\xi_{t_1}, \xi_{t_2}, \dot{\xi}_{t_1}, \dot{\xi}_{t_2}\}$. By Lemma 3 there exists t_0 such that for all t_1, t_2 for which $|t_1 - t_2| < t_0$,

$$|R_{t_2-t_1}| > c(t_2 - t_1)^8, \quad c > 0.$$

Introduce the function

$$\beta_k = \beta_k(\tau) = \max_{s > k\tau} (|r_s| + |r'_s| + |r''_s|).$$

Put

$$A_0 = \{t_1, t_2 \in \Delta_i : |t_1 - t_2| < \beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}\},$$

$$A_1 = \{t_1, t_2 \in \Delta_i : |t_1 - t_2| \geq t_0\}, \quad A_2 = (\Delta_i \times \Delta_i) \setminus (A_0 \cup A_1).$$

For sufficiently large τ , $A_0 A_1 = \emptyset$ for all $i < j - 1 < k - 2$. We split the outer integral in (10) into three, over the sets A_0, A_1, A_2 , respectively. The first integral will be of order $\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}$, as follows from the proof of Theorem 3 in (5). The second is estimated in the same way as in the proof of Lemma 2. To estimate the integral over A_2 , we use the obvious analogue of relation (6). We obtain:

$$|F_{t_1 \dots t_4} - F_{t_1 t_2 t_3 \infty}| < C \delta_{t_1 \dots t_4} \left[\min\{|R_{t_1 \dots t_4}|, |R_{t_1 t_2 t_3 \infty}|\} \right]^{-3/2},$$

where C depends only on u, r_0, r'_0 . Obviously, on A_2

$$\min\{|R_{t_1 \dots t_4}|, |R_{t_1 \dots \infty}|\} > c_1 (\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}),$$

and, since $\delta_{t_1 \dots t_4} = |R_{t_1 \dots t_4} - R_{t_1 \dots \infty}| < \beta_{k-j-1}$, it follows that

$$|F_{t_1 \dots t_4} - F_{t_1 \dots \infty}| < C_1(\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}).$$

Thus, for the integral (10) there is the estimate

$$|EN_i^{(2)} N_j X_k| < C_2(\beta_{j-i-1}^{1/13} + \beta_{k-j-1}^{1/13}).$$

Similarly one estimates $|EN_i^{(2)} X_k|$. For the other terms in (10) we easily obtain $|EN_i N_j X_k| < C_3 \beta_{k-j-1}$, $|EN_j^2 X_k| < C_4 \beta_{k-j-1}$. Finally,

$$\sum_{i < j-1 < k-2} |EX_i^2 X_j X_k| = O\left(\sum_{i < j-1 < k-2} (\beta_{j-i-1}^{1/3} + \beta_{k-j-1}^{1/3}) + n^2\right) = O(n^2).$$

For the last term in (8) we find the estimate:

$$\begin{aligned} & \sum_{i < j-1 < k-2 < l-3} |EX_i X_j X_k X_l| = \\ & = O\left(\sum_{i < j-1 < k-2 < l-3} \min(\beta_{j-i-1}, \beta_{l-k-1})\right) = O\left(n^2 \sum_0^\infty i \beta_i\right) = O(n^2). \end{aligned}$$

Thus, $\mu_4(n\tau) = O(n^2)$ for some sufficiently large τ . Hence the assertion of Lemma 4 follows.

Suppose now that ξ_t satisfies the strong mixing condition in the sense that

$$\rho(t) = \sup_{\substack{A \in \mathcal{F}_{-\infty}^0 \\ B \in \mathcal{F}_t^\infty}} |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)| \xrightarrow[t \rightarrow \infty]{} 0, \quad (11)$$

where \mathcal{F}_a^b is the σ -algebra generated by ξ_t , $t \in [a, b]$. As a consequence of Lemmas 2 and 4, the following is obtained.

Theorem. *Under the conditions of Lemma 4 and (11)*

$$\mathbf{P}\left\{\frac{N(0, T) - EN(0, T)}{\sqrt{DN(0, T)}} < z\right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx \quad (12)$$

as $T \rightarrow \infty$, for all z .

Proof. According to Theorem 18.4.1 in (3), in order to prove (12) for $T = n\tau$, where τ is fixed, it is sufficient to verify, for any integer-valued function $p = p(n)$ such that $p \rightarrow \infty$, $p = o(n)$, and any $\varepsilon > 0$, the relation

$$\frac{n}{p\sigma_n^2} \int_{|z| > \varepsilon\sigma_n} z^2 dF_p(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $\sigma_n^2 = DN(0, n\tau)$, $F_p(z) = \mathbf{P}\{N(0, p\tau) - EN(0, p\tau) < z\}$. We have

$$\frac{n}{p\sigma_n^2} \int_{|z| > \varepsilon\sigma_n} z^2 dF_p(z) < \frac{n\mu_4(p\tau)}{\varepsilon^2 p\sigma_n^4} = O\left(\frac{p}{n}\right) = o(1).$$

Now, for arbitrary T , (12) is proved with the aid of the obvious proposition: if $\mathbf{P}\{\xi_n + \eta_n < x\} \rightarrow F(x)$, where F is some distribution function and $\eta_n \rightarrow 0$ in probability, then $\mathbf{P}\{\xi_n < x\} \rightarrow F(x)$.

I take this opportunity to express my gratitude to Prof. O. V. Sarmanov for proposing the problem and for his attention to its solution.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
25 II 1967

REFERENCES

1. K. Ito, *J. Math. Kyoto Univ.*, **3**, 2, 207 (1964).
2. K. Ito, *Probability Processes*, vol. 1, IL, 1960.
3. I. A. Ibragimov, Yu. V. Linnik, *Independent and Stationarily Related Random Variables*, "Nauka," 1965.
4. H. Cramér, M. R. Leadbetter, *Ann. Math. Stat.*, **36**, 6, 1656 (1965).
5. Yu. K. Belyaev, *Theory of Probability and Its Applications*, **11**, 1, 120 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.